Linear Algebra
C Term, Sections C01-C04
W. J. Martin
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Solutions — Linear Algebra Quiz 7

(a) Find all eigenvalues of the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

Solution: We want to find all λ for which $\lambda I - A$ is singular. So we compute

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 3)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 2)(\lambda - 4)$$

So the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = 2$.

(b) Find a basis for each eigenspace of A

Solution: First, we consider $\lambda_1 = 4$. We row reduce the augmented matrix

$$[4I - A|\mathbf{0}] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solutions are $\mathbf{x} = [x_1 \ \ x_2]^T$ where $x_1 = x_2 = r$ and r can be any real number. So the eigenspace has dimension one and a basis is $T_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Next, we consider $\lambda_2 = 2$. We row reduce the augmented matrix

$$[2I - A|\mathbf{0}] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solutions are $\mathbf{x} = [x_1 \ \ x_2]^T$ given by $x_1 = -r$, $x_2 = r$ where r can be any real number. So this eigenspace also has dimension one and a basis is $T_2 = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.