Linear Algebra
C Term, Sections C01-C04
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Solutions — Linear Algebra Quiz 5

Consider the vector space P of all polynomials in t. Determine whether or not the set

$$S = \left\{ t^5 - t^3 + t, \quad 2t^4 + 2t^3 + 2t^2, \quad t^4 + 2t^3 - t, \quad 2t^5 + t^4 + t \right\}$$

of vectors in P is linearly independent in P. If S is a dependent set, express one vector as a linear combination of the others. Show all of your work.

Solution: Denote the four vectors by $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$, respectively. We would like to decide if there exist non-zero scalars $c_1, \ldots, c_4$ satisfying

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 = \mathbf{0}$$

where $\mathbf{0}$ denotes the zero polynomial. By substitution and rearrangement, we obtain

$$(c_1 + 2c_4)t^5 + (2c_2 + c_3 + c_4)t^4 + (-c_1 + 2c_2 + 2c_3)t^3 + 2c_2t^2 + (c_1 - c_3 + c_4)t = 0.$$

Two polynomials are equal precisely when all corresponding coefficients are equal. We thus obtain a linear system of five equations in four unknowns with augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So we conclude the vectors are **LINEARLY DEPENDENT**. For example, setting the free variable c_4 equal to one, we have

$$c_1 = -2, \qquad c_2 = 0, \qquad c_3 = -1, \qquad c_4 = 1.$$

So we may write

$$\mathbf{v}_4 = 2\mathbf{v}_1 + \mathbf{v}_3,$$

i.e.,

$$2t^5 + t^4 + t = 2(t^5 - t^3 + t) + (t^4 + 2t^3 - t).$$