Linear Algebra  
C Term, Sections C01-C04  
W. J. Martin  
February 11, 2002  

Solutions — Linear Algebra Quiz 5

Consider the vector space $P$ of all polynomials in $t$. Determine whether or not the set

$$S = \{t^5 - t^3 + t, \ 2t^4 + 2t^3 + 2t^2, \ t^4 + 2t^3 - t, \ 2t^5 + t^4 + t\}$$

of vectors in $P$ is linearly independent in $P$. If $S$ is a dependent set, express one vector as a linear combination of the others. Show all of your work.

**Solution:** Denote the four vectors by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, respectively. We would like to decide if there exist non-zero scalars $c_1, \ldots, c_4$ satisfying

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 + c_4 \mathbf{v}_4 = \mathbf{0}$$

where $\mathbf{0}$ denotes the zero polynomial. By substitution and rearrangement, we obtain

$$(c_1 + 2c_4)t^5 + (2c_2 + c_3 + c_4)t^4 + (-c_1 + 2c_2 + 2c_3)t^3 + 2c_2t^2 + (c_1 - c_3 + c_4)t = 0.$$

Two polynomials are equal precisely when all corresponding coefficients are equal. We thus obtain a linear system of five equations in four unknowns with augmented matrix

$$
\begin{bmatrix}
1 & 0 & 0 & 2 & 0 \\
0 & 2 & 1 & 1 & 0 \\
-1 & 2 & 2 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
1 & 0 & -1 & 1 & 0
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & 0 & 2 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

So we conclude the vectors are **LINEARLY DEPENDENT**. For example, setting the free variable $c_4$ equal to one, we have

$$c_1 = -2, \quad c_2 = 0, \quad c_3 = -1, \quad c_4 = 1.$$  

So we may write

$$\mathbf{v}_4 = 2\mathbf{v}_1 + \mathbf{v}_3,$$

i.e.,

$$2t^5 + t^4 + t = 2(t^5 - t^3 + t) + (t^4 + 2t^3 - t).$$