Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Assignment 7

1. Exercise #14 on page 304

Solution: (a) To find $[\mathbf{v}]_T$, we need to compute constants c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$$

this equation leads to the linear system whose augmented matrix is:

$$\left[\begin{array}{rrr|r} -1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & 8 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 28 \end{array}\right]$$

We have transformed the augmented matrix to reduced echelon form and we read off the solution: $c_1 = -9,\ c_2 = -8,\ c_3 = 28$. So the coordinate vector of $\mathbf{v}$ with respect to the basis T is:

$$[\mathbf{v}]_T = \begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix}$$

We find $[\mathbf{w}]_T$ by the same method: we need to compute constant c_1, c_2 and c_3 such that

$$c_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ -2 \end{bmatrix}$$

We obtain a linear system whose augmented matrix is:

$$\left[\begin{array}{rrr|r} -1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 8 \\ 0 & -1 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

So the coordinate vector of $\mathbf{w}$ with respect to the basis T is:

$$[\mathbf{w}]_T = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(b) Suppose we let:

$$\mathbf{v_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{v_2} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v_3} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{w_1} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{w_2} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \qquad \mathbf{w_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

To compute $P_{S \leftarrow T}$, we find a_1, a_2, a_3; b_1, b_2, b_3 and c_1, c_2, c_3 such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 = \mathbf{w}_1$$

$$b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 = \mathbf{w}_2$$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}_3$$

since the coefficient matrix of all three linear system is $[\mathbf{v_1} \quad \mathbf{v_2} \quad \mathbf{v_3}]$, we can transform the three augmented matrix to reduced row echelon form simultaneously by transforming the partitioned matrix:

$$\left[\begin{array}{ccc|c|c|c} 1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 & -1 & 0 \end{array}\right]$$

to reduced row echelon form, obtaining:

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & -2 & -5 & -2 \\ 0 & 1 & 0 & -1 & -6 & -2 \\ 0 & 0 & 1 & 1 & 2 & 1 \end{array}\right]$$

which implies that the transition matrix from T-basis to the S-basis is:

$$P_{S\leftarrow T} = \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

(c)

$$[\mathbf{v}]_S = P_{S\leftarrow T}[\mathbf{v}]_T = \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$[\mathbf{w}]_S = P_{S\leftarrow T}[\mathbf{w}]_T = \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ -17 \\ 8 \end{bmatrix}$$

(d) Similar to (a), to find $[\mathbf{v}]_S$, we need to compute constant c_1, c_2 and c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{v}$$

this equation leads to the linear system whose augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 1 & 0 & 2 & 8 \end{array}\right]$$

transforming the augmented matrix to reduced echelon form, we obtain the solution:

$$c_1 = 2, \qquad c_2 = 1, \qquad c_3 = 3$$

so the coordinate vector of $\mathbf{v}$ with respect to the basis T is:

$$[\mathbf{v}]_S = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

which is the same as result from (c).

To find $[\mathbf{w}]_S$, we need to compute constant c_1, c_2 and c_3 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{w}$$

this equation leads to the linear system whose augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 8 \\ 1 & 0 & 2 & -2 \end{array}\right]$$

transforming the augmented matrix to reduced echelon form, we obtain the solution:

$$c_1 = -18, \qquad c_2 = -17, \qquad c_3 = 8$$

so the coordinate vector of $\mathbf{w}$ with respect to the basis T is:

$$[\mathbf{w}]_S = \begin{bmatrix} -18 \\ -17 \\ 8 \end{bmatrix}$$

which is also the same as result from (c).

(e) By **Theorem 6.15**:

$$Q_{T\leftarrow S} = P_{S\leftarrow T}^{-1} = \begin{bmatrix} -2 & -5 & -2 \\ -1 & -6 & -2 \\ 1 & 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix}$$

(f)

$$[\mathbf{v}]_T = Q_{T\leftarrow S}[\mathbf{v}]_S = \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ -8 \\ 28 \end{bmatrix}$$

$$[\mathbf{w}]_T = Q_{T\leftarrow S}[\mathbf{w}]_S = \begin{bmatrix} -2 & 1 & -2 \\ -1 & 0 & -2 \\ 4 & -1 & 7 \end{bmatrix}\begin{bmatrix} -18 \\ -17 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

So the answers are the same as those of (a).

2 Suppose A is a 4×4 matrix with eigenvalues $4, 3, 2, 1$.
Solution: (a) Suppose λ is a eigenvalue of A, we have $A\mathbf{x} = \lambda\mathbf{x}$, Then $(5A)\mathbf{x} = 5(A\mathbf{x}) = (5\lambda)\mathbf{x}$. So 5λ is a eigenvalue of $5A$. In our case, the eigenvalues of $5A$ are 20,15,10 and 5.

(b) Suppose λ is any eigenvalue of A and $\mathbf{x}$ is an associated eigenvector. We have $A\mathbf{x} = \lambda\mathbf{x}$, Then

$$A^2\mathbf{x} = (AA)\mathbf{x} = A(A\mathbf{x} = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}.$$

So λ^2 is a eigenvalue of A^2. In our case, the eigenvalue of A^2 are 16,9,4 and 1.

(b) Since $(\lambda I - A)^T = \lambda I^T - A^T = \lambda I - A^T$, we have $det(\lambda I - A) = det(\lambda I - A^T)$. That is A and A^T have the same characteristic polynomial. So A and A^T have the same eigenvalues. In our case, the eigenvalue of A^T are 4,3,2,1.

3 Let $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$

Solution: (a) The characteristic polynomial of A is:

$$f(\lambda) = \lambda^2 - 5\lambda = \lambda(\lambda - 5)$$

The eigenvalue of A are then: $\lambda_1 = 0$, $\lambda_2 = 5$.

To find the eigenvector $\mathbf{x_1}$ associated with $\lambda_1 = 0$, we form the system

$$(0I_2 - A)\mathbf{x} = \mathbf{0}$$

or

$$\begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A solution is

$$\begin{bmatrix} -\frac{1}{2}r \\ r \end{bmatrix}$$

for any real number r. thus for $r = 2$,

$$\mathbf{x_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

is an eigenvector of A associated with $\lambda_1 = 0$.

To find the eigenvector $\mathbf{x_2}$ associated with $\lambda_2 = 5$, we form the system

$$(5I_2 - A)\mathbf{x} = \mathbf{0}$$

or

$$\begin{bmatrix} 1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A solution is

$$\begin{bmatrix} 2r \\ r \end{bmatrix}$$

for any real number r. thus for $r = 1$,

$$\mathbf{x_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is an eigenvector of A associated with $\lambda_2 = 5$.

(b) Since the eigenvectors

$$\mathbf{x_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

are linearly independent. Hence A is diagonalizable. Here

$$P = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} -0.2 & 0.4 \\ 0.4 & 0.2 \end{bmatrix}$$

Thus

$$P^{-1}AP = \begin{bmatrix} -0.2 & 0.4 \\ 0.4 & 0.2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} = D$$

(c) We see that A^5 has eigenvalues 0 and 5^5. The associated eigenvectors are the same as for A:

$$\mathbf{x_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

then we have:

$$P^{-1}A^5P = \begin{bmatrix} -0.2 & 0.4 \\ 0.4 & 0.2 \end{bmatrix} A^5 \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5^5 \end{bmatrix} = D$$

that is,

$$A^5 = PDP^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 5^5 \end{bmatrix} \begin{bmatrix} -0.2 & 0.4 \\ 0.4 & 0.2 \end{bmatrix} = \begin{bmatrix} 2500 & 1250 \\ 1250 & 625 \end{bmatrix}$$

4 Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Solution: (a) Since A is a symmetric matrix, there exists an invertible matrix P such that $P^{-1}AP = D$, where D is a diagonal matrix and the eigenvalues of A lie on the main diagonal of D. Moreover, we know that eigenvectors associated to distinct eigenvalues are orthogonal. Now we have already known 2 eigenvectors of A, we can easily calculate another eigenvector $\mathbf{v}_3 = [1 \quad 1 \quad -2]$ which is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. This gives us

$$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \qquad \text{and we have} \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

where we have denoted the remaining eigenvalue by λ. We have:

$$P^{-1} = \frac{1}{6} \begin{bmatrix} -3 & 3 & 0 \\ 2 & 2 & 2 \\ 1 & 1 & -2 \end{bmatrix}$$

and then

$$A = PDP^{-1} = \frac{1}{6} \begin{bmatrix} (-1+\lambda) & (5+\lambda) & 2(1-\lambda) \\ (5+\lambda) & (-1+\lambda) & 2(1-\lambda) \\ 2(1-\lambda) & 2(1-\lambda) & 2(1+2\lambda) \end{bmatrix}$$

This is the general form for A and, again, the remaining eigenvalue is λ.

(b) Since if A is a symmetric matrix, then eigenvectors that are associated with distinct eigenvalues of A are orthogonal. But we know that $\mathbf{v}_1$ and $\mathbf{w}_2$ are not orthogonal, so there is no such symmetric matrix.