Sample Solutions – Assignment 6

1. Find a basis for the null space of $\lambda I - A$ where $\lambda = 4$ and

$$A = \begin{bmatrix} 5 & 0 & -2 \\ -3 & 4 & 6 \\ 2 & 0 & 0 \end{bmatrix}$$

**Solution:** We form $4I_3 - A$:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & -2 \\ -3 & 4 & 6 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -6 \\ -2 & 0 & 4 \end{bmatrix}$$

This last matrix is the coefficient matrix of the homogeneous system $(\lambda I - A)x = 0$. To find all solutions, we form the augmented matrix and row reduce:

$$\begin{bmatrix} 5 & 0 & -2 & 0 \\ -3 & 4 & 6 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Every solution is then of the form: $x = \begin{bmatrix} 2s \\ \tau \\ s \end{bmatrix}$ where $\tau$ and $s$ can be any real numbers. Then every vector in the solution set can be written as:

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

So

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for the null space of $\lambda I - A$.

2. **Exercise #32 on page 293**

**Solution:** Consider the linear system:

$$\begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -13 \\ 3 \end{bmatrix}$$

We row reduce the augmented matrix to obtain

$$[A|b] \sim \begin{bmatrix} 1 & 2 & 5 & -2 & 1 \\ 0 & -1 & -12 & 8 & -15 \\ 0 & 0 & 83 & -60 & 133 \end{bmatrix}$$

after three steps. We can stop here since we can see where the “echelon” is in the reduced row echelon form. It is also clear that we can obtain a matrix row equivalent to $A$ and in row echelon form by ignoring the last column of this result. Since $\text{rank}(A) = \text{rank}([A|b]) = 3$, the linear system is guaranteed to have a solution by Theorem 6.14.
3. Solution: (a) Since the row space of $A$ and the row space of $B$ are both subspaces of the row space of $C$, it is easily seen that a lower bound of the rank of $C$ is $\max\{\text{rank}(A), \text{rank}(B)\}$. One useful upper bound on the rank of $C$ is $\text{rank}(A) + \text{rank}(B)$ since this is the total number of nonzero rows in the concatenation of their two reduced row echelon forms. So we have

$$\max\{\text{rank}(A), \text{rank}(B)\} \leq \text{rank}(C) \leq \text{rank}(A) + \text{rank}(B)$$

(b) Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Then $\text{rank}(C) = \max\{\text{rank}(A), \text{rank}(B)\} = \max\{3, 2\} = 3$.

(c) Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $\text{rank}(C) = \text{rank}(A) + \text{rank}(B) = 3 + 2 = 5$.

4. Exercise #2 on page 354
Solution: (a) To check that $x_1$ is an eigenvector associated to the given $\lambda$, we must simply verify that $Ax_1 = \lambda x_1$. So we compute

$$Ax_1 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \lambda_1 x_1$$

This shows that $\lambda_1 = -1$ is an eigenvalue of $A$ and $x_1$ is an associated eigenvector.

(b) Since

$$Ax_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 4 \end{bmatrix} = \lambda_2 x_2$$

$\lambda_2 = 2$ is an eigenvalue of $A$ and $x_2$ is an associated eigenvector.

(c) Since

$$Ax_3 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 32 \\ 20 \\ 8 \end{bmatrix} = \lambda_3 x_3$$

$\lambda_3 = 4$ is an eigenvalue of $A$ and $x_3$ is an associated eigenvector.