Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Assignment 6

1. Find a basis for the null space of $\lambda I - A$ where $\lambda = 4$ and

$$A = \begin{bmatrix} 5 & 0 & -2 \\ -3 & 4 & 6 \\ 2 & 0 & 0 \end{bmatrix}$$

Soluton: We form $4I_3 - A$:

$$4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & -2 \\ -3 & 4 & 6 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -6 \\ -2 & 0 & 4 \end{bmatrix}$$

This last matrix is the coefficient matrix of the homogeneous system $(\lambda I - A)\mathbf{x} = \mathbf{0}$. To find all solutions, we form the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 5 & 0 & -2 & 0 \\ -3 & 4 & 6 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Every solution is then of the form: $\mathbf{x} = \begin{bmatrix} 2s \\ r \\ s \end{bmatrix}$ where r and s can be any real numbers. Then every vector in the solution set can be written as:

$$\mathbf{x} = r\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

So

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for the null space of $\lambda I - A$.

2. Exercise #32 on page 293
Solution: Consider the linear system:

$$\begin{bmatrix} 1 & 2 & 5 & -2 \\ 2 & 3 & -2 & 4 \\ 5 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -13 \\ 3 \end{bmatrix}$$

We row reduce the augmented matrix to obtain

$$[A|\mathbf{b}] \sim \left[\begin{array}{cccc|c} 1 & 2 & 5 & -2 & 1 \\ 0 & -1 & -12 & 8 & -15 \\ 0 & 0 & 83 & -60 & 133 \end{array}\right] \begin{array}{l} (R1) \\ (R2) - 2\cdot(R1) \\ (R3) + 13\cdot(R1) - 9\cdot(R2) \end{array}$$

after three steps. We can stop here since we can see where the "echelon" is in the reduced row echelon form, It is also clear that we can obtain a matrix row equivalent to A and in row echelon form by ignoring the last column of this result. Since $\mathsf{rank}(A) = \mathsf{rank}([A|\mathbf{b}]) = 3$, the linear system is guaranteed to have a solution by Theorem 6.14.

3.

Solution: (a) Since the row space of A and the row space of B are both subspaces of the row space of C, it is easily seen that a lower bound of the rank of C is $\max\{\text{rank}(A), \text{rank}(B)\}$. One useful upper bound on the rank of C is $\text{rank}(A)+\text{rank}(B)$ since this is the total number of nonzero rows in the concatenation of their two reduced row echelon forms. So we have

$$\max\{\text{rank}(A), \text{rank}(B)\} \le \text{rank}(C) \le \text{rank}(A) + \text{rank}(B)$$

(b) Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Then $\text{rank}(C) = \max\{\text{rank}(A), \text{rank}(B)\} = \max\{3, 2\} = 3$.

(c) Suppose

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then $\text{rank}(C) = \text{rank}(A) + \text{rank}(B) = 3 + 2 = 5$.

4. Exercise #2 on page 354

Solution: (a) To check that $\mathbf{x}_1$ is an eigenvector associated to the given λ, we must simply verify that $A\mathbf{x}_1 = \lambda\mathbf{x}_1$. So we compute

$$A\mathbf{x}_1 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \lambda_1\mathbf{x}_1$$

This shows that $\lambda_1 = -1$ is an eigenvalue of A and $\mathbf{x}_1$ is an associated eigenvector.

(b) Since

$$A\mathbf{x}_2 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 4 \end{bmatrix} = \lambda_2\mathbf{x}_2$$

$\lambda_2 = 2$ is an eigenvalue of A and $\mathbf{x}_2$ is an associated eigenvector.

(c) Since

$$A\mathbf{x}_3 = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 32 \\ 20 \\ 8 \end{bmatrix} = \lambda_3\mathbf{x}_3$$

$\lambda_3 = 4$ is an eigenvalue of A and $\mathbf{x}_3$ is an associated eigenvector.