Linear Algebra  
C Term, Sections C01-C04  
W. J. Martin  
February 13, 2002

Sample Solutions – Assignment 5

1. Exercise #12 on page 261  
Solution: (a) Since considering \( \{t^2 + 1, t - 2, t + 3\} \) are linearly dependent or not is equivalent as considering the vectors \( \mathbf{v}_1 = (1, 0, 1) \), \( \mathbf{v}_2 = (0, 1, -2) \) and \( \mathbf{v}_3 = (0, 1, 3) \) are linearly dependent or not.  
So we form the equation:

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = 0
\]

and solve for \( c_1, c_2 \) and \( c_3 \). The resulting homogeneous system has only the trivial solution \( c_1 = c_2 = c_3 = 0 \), showing the given vectors are linearly independent. Hence \( \{t^2 + 1, t - 2, t + 3\} \) is linearly independent.

(b) Similarly, in order to know \( \{2t + 1, t^2 + 3, t\} \) are linearly dependent or not, we consider the vectors \( \mathbf{v}_1 = (2, 0, 1) \), \( \mathbf{v}_2 = (1, 0, 3) \) and \( \mathbf{v}_3 = (0, 1, 0) \) are linearly dependent or not.  
So we form the equation:

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = 0
\]

and solve for \( c_1, c_2 \) and \( c_3 \). Also the resulting homogeneous system has only the trivial solution \( c_1 = c_2 = c_3 = 0 \), showing that the vectors are linearly independent. Hence \( \{2t^2 + 1, t^2 + 3, t\} \) is linearly independent.

(c) Given \( \{3t + 1, 3t^2 + 1, 2t^2 + t + 1\} \), we consider the vectors \( \mathbf{v}_1 = (0, 3, 1) \), \( \mathbf{v}_2 = (1, 0, 3) \) and \( \mathbf{v}_3 = (2, 1, 1) \) are linearly dependent or not.  
So we form the equation:

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = 0
\]

and solve for \( c_1, c_2 \) and \( c_3 \). The resulting homogeneous system has infinitely many solutions. A particular solution is \( c_1 = \frac{1}{3}, c_2 = \frac{2}{3}, c_3 = -1 \). Or we can write \( 2t^2 + t + 1 = \frac{1}{3}(3t + 1) + \frac{2}{3}(3t^2 + 1) \). Hence \( \{2t^2 + 1, t^2 + 3, t\} \) is linearly dependent.

(d) Given \( \{t^2 - 4, 5t^2 - 5t - 6, 3t^2 - 5t + 2\} \), we consider the vectors \( \mathbf{v}_1 = (1, 0, -4) \), \( \mathbf{v}_2 = (5, -5, 6) \) and \( \mathbf{v}_3 = (3, -5, 2) \) are linearly dependent or not.  
So we form the equation:

\[
c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 = 0
\]

and solve for \( c_1, c_2 \) and \( c_3 \). The resulting homogeneous system has only the trivial solution \( c_1 = c_2 = c_3 = 0 \), showing that the vectors are linearly independent. Hence \( \{t^2 - 4, 5t^2 - 5t - 6, 3t^2 - 5t + 2\} \) is linearly independent.

2. Exercise #14 on page 262  
Solution: (a) Since \( \sin t \) can’t be written as \( \sin t = k \cos t \) and the unbounded function \( e^t \) can’t be written as the linear combination of bounded functions \( \sin t \) and \( \cos t \). So by Theorem 6.4, \( \{\cos t, \sin t, e^t\} \) is linearly independent.

(b) Since \( e^t \) can’t be written as \( e^t = kt \) and the bounded function \( \sin t \) can’t
be written as the linear combination of unbounded functions \( t \) and \( e^t \). So by Theorem 6.4, \( \{t, e^t, \sin t\} \) is linearly independent.

(c) Since \( t \) can't be written as \( t = kt^2 \) and the function \( e^t \) can't be written as the linear combination of functions \( t^2 \) and \( t \). So by Theorem 6.4, \( \{t^2, t, e^t\} \) is linearly independent.

(d) Since \( \cos 2t \) can be written as \( \cos^2 t - \sin^2 t \). So \( \{\cos^2 t, \sin^2 t, \cos 2t\} \) is linearly dependent.

3. Exercise #14 on page 273

Solution: Step 1. Since
\[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 1
\end{bmatrix}
\]
We can delete \( \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix} \) from \( S \), getting the subset \( S_1=\left\{ \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
1 & 0 \\
1 & 1
\end{bmatrix}, \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix} \right\} \), which also spans \( W \).

Step 2. Since
\[
\begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}
= (-1) \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
+ \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]
We can delete \( \begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix} \) from \( S_1 \), getting the subset \( S_2=\left\{ \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \right\} \), which also spans \( W \).

Step 3. Since now \( S_2=\left\{ \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 1
\end{bmatrix}, \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \right\} \) spans \( W \) and is linearly independent. Thus \( S_2 \) is a basis for \( W \).

4. Exercise #T.10 on page 274

Since the number of vectors in \( T \) equals the dimension of \( V \), we have two ways to prove \( T \) is a basis for \( V \).

Proof(1): To prove \( T = \{w_1, w_2, w_3\} \) spans \( V \).

Since \( S = \{v_1, v_2, v_3\} \) is a basis for vector space \( V \). So for any vector \( v \) in \( V \), there is \( c_1, c_2 \) and \( c_3 \) such that \( v = c_1 v_1 + c_2 v_2 + c_3 v_3 \). Let \( k_1 = c_1, k_2 = c_2 - c_1, k_3 = c_3 - c_2 \), we can see that
\[
k_1 w_1 + k_2 w_2 + k_3 w_3 = c_1 (v_1 + v_2 + v_3) + (c_2 - c_1) (v_2 + v_3) + (c_3 - c_2) v_3
= c_1 v_1 + c_2 v_2 + c_3 v_3 = v.
\]
So \( T = \{w_1, w_2, w_3\} \) spans \( V \) and is a basis for \( V \).

Proof(2): To prove \( T = \{w_1, w_2, w_3\} \) is linearly independent.

Suppose there is \( k_1, k_2, k_3 \) such that \( k_1 w_1 + k_2 w_2 + k_3 w_3 = 0 \) and \( k_1, k_2, k_3 \) not all equal to 0. Then we have:
\[
k_1 w_1 + k_2 w_2 + k_3 w_3 = k_1 (v_1 + v_2 + v_3) + k_2 (v_2 + v_3) + k_3 v_3
= k_1 v_1 + (k_1 + k_2) v_2 + (k_1 + k_2 + k_3) v_3 = 0
\]
Since \( k_1, k_2, k_3 \) not all equal to 0, so \( k_1, (k_1 + k_2) \) and \( (k_1 + k_2 + k_3) \) can't all equal to 0. But this contradicts the fact that \( S = \{v_1, v_2, v_3\} \) is linearly independent. That is, such \( k_1, k_2 \) and \( k_3 \) don't exist. So \( T = \{w_1, w_2, w_3\} \) is linearly independent and is a basis for \( V \).