Linear Algebra
C Term, Sections C01-C04
W. J. Martin
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Sample Solutions – Assignment 5

1. Exercise #12 on page 261

Solution: (a) Since considering $\{t^2+1, t-2, t+3\}$ are linearly dependent or not is equivalent as considering the vectors $\mathbf{v_1} = (1,0,1)$, $\mathbf{v_2} = (0,1,-2)$ and $\mathbf{v_3} = (0,1,3)$ are linearly dependent or not.
So we form the equation:

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$$

and solve for c_1, c_2 and c_3. The resulting homogeneous system has only the trivial solution $c_1 = c_2 = c_3 = 0$, showing the given vectors are linearly independent. Hence $\{t^2+1, t-2, t+3\}$ is linearly independent.

(b) Similarly, in order to know $\{2t^2+1, t^2+3, t\}$ are linearly dependent or not, we consider the vectors $\mathbf{v_1} = (2,0,1)$, $\mathbf{v_2} = (1,0,3)$ and $\mathbf{v_3} = (0,1,0)$ are linearly dependent or not.
So we form the equation:

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$$

and solve for c_1, c_2 and c_3. Also the resulting homogeneous system has only the trivial solution $c_1 = c_2 = c_3 = 0$, showing that the vectors are linearly independent. Hence $\{2t^2+1, t^2+3, t\}$ is linearly independent.

(c) Given $\{3t+1, 3t^2+1, 2t^2+t+1\}$, we consider the vectors $\mathbf{v_1} = (0,3,1)$, $\mathbf{v_2} = (3,0,1)$ and $\mathbf{v_3} = (2,1,1)$ are linearly dependent or not.
So we form the equation:

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$$

and solve for c_1, c_2 and c_3. The resulting homogeneous system has infinitely many solutions. A particular solution is $c_1 = \frac{1}{3}, c_2 = \frac{2}{3}, c_3 = -1$. Or we can write $2t^2+t+1 = \frac{1}{3}(3t+1) + \frac{2}{3}(3t^2+1)$. Hence $\{2t^2+1, t^2+3, t\}$ is linearly dependent.

(d) Given $\{t^2-4, 5t^2-5t-6, 3t^2-5t+2\}$, we consider the vectors $\mathbf{v_1} = (1,0,-4)$, $\mathbf{v_2} = (5,-5,6)$ and $\mathbf{v_3} = (3,-5,2)$ are linearly dependent or not.
So we form the equation:

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{0}$$

and solve for c_1, c_2 and c_3. The resulting homogeneous system has only the trivial solution $c_1 = c_2 = c_3 = 0$, showing that the vectors are linearly independent. Hence $\{t^2-4, 5t^2-5t-6, 3t^2-5t+2\}$ is linearly independent.

2. Exercise #14 on page 262

Solution: (a) Since $\sin t$ can't be written as $\sin t = k\cos t$ and the unbounded function e^t can't be written as the linear combination of bounded functions $\sin t$ and $\cos t$. So by Theorem 6.4, $\{\cos t, \sin t, e^t\}$ is linearly independent.

(b) Since e^t can't be written as $e^t = kt$ and the bounded function $\sin t$ can't be written as the linear combination of unbounded functions t and e^t. So by Theorem 6.4, $\{t, e^t, \sin t\}$ is linearly independent.

(c) Since t can't be written as $t = kt^2$ and the function e^t can't be written as the linear combination of functions t^2 and t. So by Theorem 6.4, $\{t^2, t, e^2\}$ is linearly independent.

(d) Since $\cos 2t$ can be written as $\cos^2 t - \sin^2 t$. So $\{\cos^2 t, \sin^2 t, \cos 2t\}$ is linearly dependent.

3. Exercise #14 on page 273

Solution: Step 1. Since $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ We can delete $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ from S, getting the subset $S_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$, which also spans W

Step 2. Since $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = (-1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
We can delete $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ from S_1, getting the subset $S_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$, which also spans W

Step 3. Since now $S_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ spans W and is linearly independent. Thus S_2 is a basis for W.

4. Exercise #T.10 on page 274

Since the number of vectors in T equals the dimension of V, We have two ways to prove T is a basis for V.
Proof(1): To prove $T = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ spans V.
Since $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a basis for vector space V. So for any vector $\mathbf{v}$ is V, there is c_1, c_2 and c_3 such that $\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3}$. Let $k_1 = c_1, k_2 = c_2 - c_1, k_3 = c_3 - c_2$, we can see that

$$\begin{aligned} k_1\mathbf{w_1} + k_2\mathbf{w_2} + k_3\mathbf{w_3} &= c_1(\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}) + (c_2 - c_1)(\mathbf{v_2} + \mathbf{v_3}) + (c_3 - c_2)\mathbf{v_3} \\ &= c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = \mathbf{v}. \end{aligned}$$

So $T = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ spans V and is a basis for V

Proof(2): To prove $T = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is linearly independent.
Suppose there is k_1, k_2, k_3 such that $k_1\mathbf{w_1} + k_2\mathbf{w_2} + k_3\mathbf{w_3} = \mathbf{0}$ and k_1, k_2, k_3 not all equal to 0. Then we have:

$$\begin{aligned} k_1\mathbf{w_1} + k_2\mathbf{w_2} + k_3\mathbf{w_3} &= k_1(\mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}) + k_2(\mathbf{v_2} + \mathbf{v_3}) + k_3\mathbf{v_3} \\ &= k_1\mathbf{v_1} + (k_1 + k_2)\mathbf{v_2} + (k_1 + k_2 + k_3)\mathbf{v_3} = \mathbf{0} \end{aligned}$$

Since k_1, k_2, k_3 not all equal to 0, so k_1,$(k_1 + k_2)$ and $(k_1 + k_2 + k_3)$ can't all equal to 0. But this contradicts the fact that $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is linearly independent. That is, such k_1, k_2 and k_3 don't exist. So $T = \{\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}\}$ is linearly independent and is a basis for $\mathbf{V}$.