Linear Algebra
C Term, Sections C01-C04
W. J. Martin
February 6, 2002

Sample Solutions – Assignment 4

1. Exercise #22 on page 236

Soluton: Since the desired plane is parallel with the plane $-2x+4y-5z+6=0$, so it has a normal $\mathbf{n}=(-2,4,-5)$. Also we know that it passes through the point $(2,4,-3)$. Thus, an equation for the desired plane is:

$$-2(x-2)+4(y-4)-5(z+3)=0$$

Simplifyng this, we have the equation

$$-2x+4y-5z-27=0$$

2. Exercise #6 on page 250

Solution: 6(a) Since $a_1=0$ and $a_0=0$, every polynomial in the set has the form $p(t)=at^2$ where a is some real number. Consider $p(t)=at^2$ and $q(t)=bt^2$. Then $p(t)+q(t)=(a+b)t^2$ is in the set since its linear and constant terms have coefficient zero. Also, if k is a scalar, then $kp(t)=(ka)t^2$ is in the same set. Hence the set of all polynomials $a_2t^2+a_1t+a_0$ having $a_1=a_0=0$ is a subspave of P_2.

6(b) Since $a_1=2a_0$, we may write any polynomial $p(t)$ in the set as $p(t)=a_2t^2+2a_0t+a_0$. Consider $p(t)=2_2t^2+2a_0t+a_0$ and $q(t)=b_2t^2+2b_0t+b_0$. Then $p(t)+q(t)=(a_2+b_2)t^2+2(a_0+b_0)t+(a_0+b_0)$ is in the same set as its coefficients clearly satisfy the same relationship. Also, if k is a scalar, then $kp(t)=(ka_2)t^2+2(ka_0)t+ka_0$ is in the specified subset. Hence the set of all polynomials (of degree at most 2) having $a_1=2a_0$ is a subspace of P_2.

6(c) This is **not** a subspace. Consider for example the polynomials

$$p(t)=t^2+2t-1, \qquad q(t)=t+1.$$

Then both $p(t)$ and $q(t)$ satisfy the given condition as

$$(1+2-1)=2, \qquad (0+1+1)=2.$$

But $p(t)+q(t)=t^2+3t$ is not in the set since its coefficients sum to four. This shows that the given set of polynomials is not closed under addition.

3. Exercise #12 on page 250

Solution: 12(a)

$$\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}, \text{ where } b=a+c$$

Consider

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 & c_1+c_2 \\ d_1+d_2 & 0 & 0 \end{bmatrix}$$

Since $b_1=a_1+c_1$ and $b_2=a_2+c_2$, so $(b_1+b_2)=(a_1+a_2)+(c_1+c_2)$.
Also, if k is a scalar,

$$k\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix} = \begin{bmatrix} ka & kb & kc \\ kd & 0 & 0 \end{bmatrix}$$

Then $kb = ka + kc$. Hence, the set of all matrices of the form

$$\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix} \text{ where } b = a + c$$

is a subspace of M_{23}

12(b) This set **fails** to be a subspace of M_{23}. Consider the vector

$$\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \text{ where } c = 1 > 0$$

Then, with scalar $k = -1$, we have

$$k\mathbf{u} = (-1)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

with $(1, 3)$-entry being negative. So $k\mathbf{u}$ is not inside the specified set. Thus the set fails to be closed under scalar multiplication. and is not a subspace of M_{23}

12(c)

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}, \text{ where } a = -2c \text{ and } f = 2e + d$$

Let $\mathbf{u}$ and $\mathbf{v}$ be any vectors of this form, say Consider

$$\mathbf{u} = \begin{bmatrix} a_1 & b_1 & c_1 \\ d_1 & e_1 & f_1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} a_2 & b_2 & c_2 \\ d_2 & e_2 & f_2 \end{bmatrix}.$$

Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & c_1 + c_2 \\ d_1 + d_2 & e_1 + e_2 & f_1 + f_2 \end{bmatrix}$$

Since $a_1 = -2c_1$ and $a_2 = -2c_2$, so $(a_1 + a_2) = -2(c_1 + c_2)$.
Since $f_1 = 2e_1 + d_1$ and $f_2 = 2e_2 + d_2$, so $(f_1 + f_2) = 2(e_1 + e_2) + (d_1 + d_2)$
So the vector $\mathbf{u} + \mathbf{v}$ does indeed belong to the set.

Also, if k is any scalar,

$$k\mathbf{u} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ kd_1 & ke_1 & kf_1 \end{bmatrix}$$

Observe that, using what we know about $\mathbf{u}$, we have $ka_1 = k(-2c_1) = -2(kc_1)$ and $kf_1 = k(2e_1 + d_1) = 2(ke_1) + kd_1$. This shows that the set of all such matrices is closed under scalar multiplication. Now we may conclude that this is a subspace.

4. Exercise #T.10 on page 252

Proof: Clearly $W_1 + W_2$ is non-empty as it contains $\mathbf{0} = \mathbf{0} + \mathbf{0}$, noting that $\mathbf{0}$ belongs to both W_1 and W_2. Let $\mathbf{u}$,$\mathbf{v}$ be any vectors in $W_1 + W_2$. Then there exist $\mathbf{u}_1$ and $\mathbf{v}_1$ in W_1 and $\mathbf{u}_2$ and $\mathbf{v}_2$ in W_2 for which

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 \qquad \text{and} \qquad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

using the definition of $W_1 + W_2$. Then we have

$$\mathbf{u} + \mathbf{v} = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2)$$

where $\mathbf{u}_1 + \mathbf{v}_1$ belongs to W_1 and $\mathbf{u}_2 + \mathbf{v}_2$ belongs to W_2 since W_1 and W_2 are subspaces. This shows that $W_1 + W_2$ is closed under vector addition. Similarly, if k is any scalar,

$$k\mathbf{u} = k\mathbf{u}_1 + k\mathbf{u}_2$$

belongs to $W_1 + W_2$ since $k\mathbf{u}_1$ belongs to W_1 and $k\mathbf{u}_2$ belongs to W_2 (these latter two sets being closed under scalar multiplication). Since $W_1 + W_2$ is non-empty and closed under both addition and scalar multiplication, we conclude that it is a subspace of V.