Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Assignment 3

1(a) Using elementary row operation, row reduce the following matrix to upper triangular form:

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ -1 & 0 & 4 & 2 & -2 \\ 2 & 6 & 5 & 2 & 4 \\ 0 & -2 & 0 & 8 & 13 \\ 1 & 4 & 4 & -3 & 2 \end{pmatrix}$$

Solution: We have:

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ -1 & 0 & 4 & 2 & -2 \\ 2 & 6 & 5 & 2 & 4 \\ 0 & -2 & 0 & 8 & 13 \\ 1 & 4 & 4 & -3 & 2 \end{pmatrix}_{\mathbf{r_1+r_2\to r_2}} \sim \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 2 & 6 & 5 & 2 & 4 \\ 0 & -2 & 0 & 8 & 13 \\ 1 & 4 & 4 & -3 & 2 \end{pmatrix}_{\mathbf{-2r_1+r_3\to r_3}} \sim$$

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & 2 & 5 & 4 & 2 \\ 0 & -2 & 0 & 8 & 13 \\ 1 & 4 & 4 & -3 & 2 \end{pmatrix}_{\mathbf{-1r_1+r_5\to r_5}} \sim \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & 2 & 5 & 4 & 2 \\ 0 & -2 & 0 & 8 & 13 \\ 0 & 2 & 4 & -2 & 1 \end{pmatrix}_{\mathbf{-1r_2+r_3\to r_3}} \sim$$

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & -2 & 0 & 8 & 13 \\ 0 & 2 & 4 & -2 & 1 \end{pmatrix}_{\mathbf{r_2+r_4\to r_4}} \sim \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 4 & 9 & 12 \\ 0 & 2 & 4 & -2 & 1 \end{pmatrix}_{\mathbf{-1r_2+r_5\to r_5}} \sim$$

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 4 & 9 & 12 \\ 0 & 0 & 0 & -3 & 2 \end{pmatrix}_{\mathbf{-4r_3+r_4\to r_4}} \sim \begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 & 2 \end{pmatrix}_{\mathbf{-1r_4+r_5\to r_5}} \sim$$

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & -1 \\ 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} = B$$

1(b) By **Theorem 3.7**, $\det(B) = 1 * 2 * 1 * -3 * 2 = -12$. Then By **Theorem 3.6**, we know that $\det(A) = \det(B) = -12$.

2. Complete Exercise 14 on page 212
(a) Solution: In order to find out whether the vector

$$\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

lies in range $\mathbf{L}$, we need to determine whether there is a vector

$$\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

so that $L(\mathbf{u}) = \mathbf{w}$. We have

$$\mathrm{L}(\mathbf{u}) = L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

The augmented matrix of this linear system can be reduced to:

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The form of the third row of this reduced matrix implies that this linear system has no solution. Hence, $\mathbf{w}$ is **not** in the range of $\mathbf{L}$.

(b) Solution: Like **(a)**, we look at the linear system:

$$\mathrm{L}(\mathbf{u}) = L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

The augmented matrix of this linear system can be reduced to:

$$\begin{pmatrix} 1 & 0 & \frac{2}{3} & \frac{5}{3} \\ 0 & 1 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Also from the form of this reduced matrix, we know that this linear system has infinitely many solutions. Hence, $\mathbf{w}$ is in the range of $\mathbf{L}$.

3. Complete Exercise 16 on page 213
Solution: In order to find an equation relating a, b, and c so that

$$\mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

will lie in range **L**, we need to make sure the following linear system has at least one solution.

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{pmatrix} 1 & 2 & 3 \\ -3 & -2 & -1 \\ -2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The augmented matrix of this linear system can be reduced to:

$$\begin{pmatrix} 1 & 0 & -1 & -\frac{1}{2}a - \frac{1}{2}b \\ 0 & 1 & 2 & \frac{3}{4}a + \frac{1}{4}b \\ 0 & 0 & 0 & -a - b + c \end{pmatrix}$$

From the third row of this reduced matrix, we know that in order to make this linear system consistent, $(-a - b + c)$ must be zero. That is, $c = a + b$. So this equation makes sure that **w** will lie in range **L**.

4. Complete Exercise 18 on page 213

Solution: Since $L : R^3 \to R^3$ is a linear transformation such that:

$$L(\mathbf{i}) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, L(\mathbf{j}) = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \text{and } L(\mathbf{k}) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

By **Theorem 4.8**, we know that there exists a unique $n \times n$ matrix such that $L(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x}$ is R^3. In fact, this standard matrix representing L is:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix}$$

Thus we have

$$L\left(\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 5 \end{pmatrix}$$