Linear Algebra
C Term, Sections C01-C04
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Sample Solutions – Assignment 2

1(a) Solution: We let x_1, x_2 and x_3 denote the number the units we will produce for the P1 P2 and P3 model, respectively. Then for model P1, we will use x_1 units of steel and x_1 units of plastic. For model P2, we will use $2x_2$ units of steel and x_2 units of plastic. For model P3, we will use $2x_3$ units of steel and $3x_3$ units of plastic. Our goal is to fully utilize the 1200 units of steel and 1000 units of plastic. Summarizing them, we then have the following system of equations:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 1200 \\ x_1 + x_2 + 3x_3 &= 1000 \end{aligned}$$

The augmented matrix of this linear system is:

$$\begin{pmatrix} 1 & 2 & 2 & 1200 \\ 1 & 1 & 3 & 1000 \end{pmatrix}$$

We row reduce to obtain the following matrix in reduced row echelon form:

$$\begin{pmatrix} 1 & 0 & 4 & 800 \\ 0 & 1 & -1 & 200 \end{pmatrix}$$

This augmented matrix corresponds to the equivalent linear system:

$$\begin{aligned} x_1 + 4x_3 &= 800 \\ x_2 - x_3 &= 200 \end{aligned}$$

So we have infinitely many solutions with one free variable. Letting $x_3 = r$ (any real number), we obtain the general form of the solution to this linear system:

$$x_1 = 800 - 4r \qquad x_2 = 200 + r \qquad x_3 = r$$

Considering the practical application, the only sensible values for r will be $r = 0, 1, 2, ..., 200$.

1(b) Solution: Since we make \$12 for each P1, \$22 for each P2 and \$30 for each P3, then the profit for a given choice of r is

$$12(800 - 4r) + 22(200 + r) + 30r = 14000 + 4r.$$

So obviously, $r = 200$ will maximize the profit. That is, optimal profit=\$14800

Production Schedule:

Model	Number to make	Steel Used	Plastic Used	Profit
P1	0	0	0	\$0
P2	400	800	400	\$8,800
P3	200	400	600	\$6,000
Total	600	1200	1000	\$14,800

2(a) Solution: We compute the determinant of A:

$$\mathbf{det}(A) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r & 2 \\ 0 & 2 & r \end{vmatrix} = r^2 - 4$$

We know that A is singular if and only if the determinant is zero. So A is singular if and only if $r = \pm 2$.

2(b) Solution: Provided $r \neq \pm 2$, we can transform the matrix $[A \vdots I_3]$ to reduced row echelon form by taking the following steps:

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & r & 2 & 0 & 1 & 0 \\ 0 & 2 & r & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{r} & 0 & \frac{1}{r} & 0 \\ 0 & 2 & r & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{2}{r} & 0 & \frac{1}{r} & 0 \\ 0 & 0 & r - \frac{4}{r} & 0 & \frac{-2}{r} & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{r}{r^2-4} & \frac{-2}{r^2-4} \\ 0 & 0 & r - \frac{4}{r} & 0 & \frac{-2}{r} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{r}{r^2-4} & \frac{-2}{r^2-4} \\ 0 & 0 & 1 & 0 & \frac{-2}{r^2-4} & \frac{r}{r^2-4} \end{pmatrix}$$

$$\text{So,} \qquad A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{r}{r^2-4} & \frac{-2}{r^2-4} \\ 0 & \frac{-2}{r^2-4} & \frac{r}{r^2-4} \end{pmatrix} = \frac{1}{r^2-4} \begin{pmatrix} r^2-4 & 0 & 0 \\ 0 & r & -2 \\ 0 & -2 & r \end{pmatrix}$$

3(a) Solution: We notice the special structure of these matrices. Matrix A is an upper triangular matrix (p16) with all entries on main diagonal equaling 1. It is easy to show that the product of two upper triangular matrix with all entries on main diagonal equaling 1 is a matrix with the same structure. So here, we guess that some square root of A has the same structure as A.

Suppose a square root of matrix $\begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}$ is $\mathrm{B} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ which satisfies $\mathrm{B}^2 = \mathrm{A}$

Then we have $x + x = 9$. i.e. $x = \dfrac{9}{2}$. So a square root of A is $\begin{pmatrix} 1 & 4.5 \\ 0 & 1 \end{pmatrix}$

3(b) Solution: Suppose a square root of A is:

$$B = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{satisfying} \qquad B^2 = \begin{pmatrix} 1 & 4 & 14 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we have the following equations: $x + x = 4$; $y + xz + y = 14$; $z + z = 8$. That is $x = 2$; $y = 3$; $z = 4$.

$$\text{So} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

3(c) Solution: We can easily list 3 square roots of I_3:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

There are infinitely many other square roots of I, so this is not the only correct answer.

4. Problem T.13 on page 65.

Throughout, assume that A and $\mathbf{b} \neq \mathbf{0}$ are given and the system $A\mathbf{x} = \mathbf{b}$ is consistent.

(a) Proof: Assume that $\mathbf{x}_p$ is some particular solution to the above sytem and that $\mathbf{x}_h$ is a solution to the system $A\mathbf{x} = \mathbf{0}$. So $A\mathbf{x}_p = \mathbf{b}$ and $A\mathbf{x}_h = \mathbf{0}$. Then

$$A(\mathbf{x}_p + \mathbf{x}_h) = A\mathbf{x}_p + A\mathbf{x}_h = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

So $\mathbf{x}_p + \mathbf{x}_h$ is a solution to the given system.

4(b) Proof: Let $\mathbf{x}_p$ be given. That is, $\mathbf{x}_p$ is any solution to the system $A\mathbf{x} = \mathbf{b}$, but it is fixed beforehand.

Now let $\mathbf{x}$ be any solution to the original system. Then we obviously have $A\mathbf{x} = \mathbf{b}$. Define

$$\mathbf{x}_h = \mathbf{x} - \mathbf{x}_p.$$

Then

$$A\mathbf{x}_h = A(\mathbf{x} - \mathbf{x}_p) = A\mathbf{x} - A\mathbf{x}_p = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

This shows that $\mathbf{x}_h$ is a solution to the associated homogeneous system. Moreover, we have $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ where $\mathbf{x}_p$ is the particular solution given in advance.

Upshot: If we obtain a method to find **all** solutions to a homogeneous system, then we can apply this "technology" to arbitrary systems. All we need to do in those cases is find just <u>one</u> solution.