Linear Algebra
C Term, Sections C01-C04
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January 16, 2002

Sample Solutions – Assignment 1

1 Problem #28 on page 9

An inheritance of \$24,000 is to be divided among three trusts, with the second trust receiving twice as much as the first trust. The three trusts pay interest at the rates of 9%, 10% and 6% annualy, respectively, and return a total in interest of \$2210 at the end of the first year. How much was invested in each trust?

Solution: We let x, y and z denote the amount the money going into the first, the second and the third trust, respectively. Then we have the following system of equations:

$$x + y + z = 24000 \tag{1a}$$
$$2x - y = 0 \tag{1b}$$
$$0.09x + 0.1y + 0.06z = 2210 \tag{1c}$$

Before we apply the method of elimination, we can eliminate y from the first and third equations to obtain

$$3x + z = 24000 \tag{2a}$$
$$0.29x + 0.06z = 2210 \tag{2b}$$

Now we subtract 0.06 times the first equation from the second to obtain $0.11x = 770$. Now we have $x = 7000$ and by substituting back into the other equations we find that the linear system has a unique solution. i.e. x=7000; y=14000; z=3000

So the amounts invested in the first, second and third trusts were \$7000, \$14000 and \$3000, respectively.

2 Problem #6 on page 29

$$\textbf{Given :} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 & -3 \\ 4 & -0 & -2 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ -1 & 5 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 3 & -1 \\ 3 & -4 & 5 \\ 1 & -1 & -2 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \quad \mathbf{E} = \begin{pmatrix} 1 & 0 & -3 \\ -2 & 1 & 5 \\ 3 & 4 & 2 \end{pmatrix} \quad \mathbf{F} = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$$

If possible, compute:

(a) A(BD). (b) (AB)D. (c) A(C+E). (d) AC+AE. (e) (D+F)A.

Solution:

$$(a) \qquad \mathbf{A}(\mathbf{BD}) = \begin{pmatrix} 1 & 2 & -3 \\ 4 & -0 & -2 \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 0 & -2 \\ -7 & -13 \end{pmatrix} = \begin{pmatrix} 26 & 42 \\ 34 & 54 \end{pmatrix}$$

$$(b) \qquad (\mathbf{AB})\mathbf{D} = \begin{pmatrix} 10 & -6 \\ 14 & -6 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 26 & 42 \\ 34 & 54 \end{pmatrix}$$

This is the same result as in part (a) since matrix multiplication is associative.

$$(c) \qquad \mathbf{A}(\mathbf{C}+\mathbf{E}) = \begin{pmatrix} 1 & 2 & -3 \\ 4 & 0 & -2 \end{pmatrix} \begin{pmatrix} 3 & 3 & -2 \\ 1 & -3 & 10 \\ 4 & 3 & 0 \end{pmatrix} = \begin{pmatrix} -7 & -12 & 18 \\ 4 & 6 & -8 \end{pmatrix}$$

$$(d) \qquad \mathbf{AC}+\mathbf{AE} = \begin{pmatrix} 5 & -2 & 17 \\ 6 & 14 & 8 \end{pmatrix} \begin{pmatrix} -12 & -10 & 1 \\ -2 & -8 & -16 \end{pmatrix} = \begin{pmatrix} -7 & -12 & 18 \\ 4 & 6 & -8 \end{pmatrix}$$

This is the same result as part (c) since matrix multiplication is distributive.

$$(e) \qquad (\mathbf{D}+\mathbf{F})\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 4 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 8 & -12 \\ -1 & 6 & -7 \end{pmatrix}$$

We note that $DA + FA$ will give the same matrix.

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(a) Find all values r and s for which

$$\begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{commutes with} \quad \begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

Solution: We compute both AB and BA and set the two matrices equal to one another:

$$AB = \begin{pmatrix} 3r & r & r \\ s & 4s & s \\ 2 & 2 & 6 \end{pmatrix} \qquad BA = \begin{pmatrix} 3r & s & 2 \\ r & 4s & 2 \\ r & s & 6 \end{pmatrix}$$

The equation $AB = BA$ gives us a system of nine linear equations in two unknowns. The interesting equations among these are

$$r = s, \qquad r = 2, \qquad s = 2.$$

So, obviously, $r = s = 2$ is the only solution. I.e., A is forced to be a multiple of the identity matrix.

(b) Find a value of r for which the matrix

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & r \end{pmatrix}$$

squares to the identity.

Solution: We compute

$$\mathbf{C}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}r \\ 0 & \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}r & \frac{3}{4} + r^2 \end{pmatrix}$$

and the matrix equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}r \\ 0 & \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}r & \frac{3}{4} + r^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

gives us two equations:

$$\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}r = 0, \qquad \frac{3}{4} + r^2 = 1.$$

We then find that $r = -\frac{1}{2}$ is the only solution.

4 Problem T.18 on page 42.

Show that if A is symmetric, then A^T is symmetric.

Proof: (We are to assume that A is symmetric and must deduce that A^T is also symmetric.)

Assume that A is symmetric. Then we have $A = A^T$. Taking the transpose of both sides, we find $A^T = (A^T)^T$. Then by definition, A^T is symmetric.