Bridge to Higher Math
D Term 2011
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MA197X Problem Set 6

Instructions: Please review the rules on the presentation of assignments in the course. Then complete the following ten problems and submit the solutions, inside your portfolio folder, by Thursday, April 28th.

For each of the following problems, first state the problem precisely in English and then give a proper proof of the statement using English sentences. Be sure to include the correct problem numbers for recording purposes.

51. For all positive integers n

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

[HINT: You may want to use a proof by contradiction in your induction step.]

52. For all positive integers n

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

53. For every positive integer n,

$$1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n} = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right).$$

54. For every positive integer n

$$2\left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) > 2\sqrt{n+1}.$$

55. For every positive integer n

$$2\left(1 + \frac{1}{8} + \cdots + \frac{1}{n^3}\right) < 3 - \frac{1}{n^2}.$$

In the remaining problems (except the last one), you need to find the theorem before you search for its proof. Using experimentation with small values of n, first make a conjecture regarding the outcome for general positive integers n and then prove your conjecture using induction. (NOTE: The experimentation should be done on scrap paper and is not part of your formal solution.)

56. Find a simple expression for the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}.$$

57. For a real number a which is not an integer, simplify

$$\frac{1}{a \cdot (a+1)} + \frac{1}{(a+1) \cdot (a+2)} + \cdots + \frac{1}{(a+n-1) \cdot (a+n)}.$$

58. What is the largest integer that divides $n^3 + (n+1)^3 + (n+2)^3$ for every positive integer n?

59. Let us call a positive integer n "posh" if it is possible to dissect a square into n smaller squares. For example, $n = 6$ and $n = 7$ are posh while $n = 1$ and $n = 2$ are not. Which natural numbers are posh?

60. The Fibonacci numbers[1] are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. Find the value of $F_{n-1}F_{n+1} - F_n^2$ for $n \geq 2$.

And here are some fun induction and Pigeonhole Principle problems to think about. Do not turn these in:

61. Simplify $1 + 2 + 4 + \cdots + 2^n$.

62. For all positive integers n,

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

63. For every positive integer n, the integer $n^3 + 5n$ is divisible by six.

64. If n lines are drawn in the plane so that no two are parallel and no three share a common point, into how many regions is the plane divided by these lines?

[1] These numbers were introduced in Year 1202 by Leonardo of Pisa, the son of Bonacci.

65. Find a relation for $F_n^2 + F_{n+1}^2$ which holds for all positive integers n.

66. If $m|n$, then F_m divides F_n. *[HINT: Use induction on $k = n/m$.]*

67. Every positive integer can be expressed as a sum of distinct Fibonacci numbers.

68. For every integer $n \geq 4$, $n! > 2^n$. *[NOTE: This requires a straightforward variation on the Principle of Mathematical Induction. Please state it — without proof — in your "Recall" section.]*

69. Prove that, if five points are chosen from the interior of an equilateral triangle of side length two, then some pair of these points lie at a distance of one or less.

70. Prove that, if 51 distinct integers $n_1, \ldots, n_{51}$ are chosen from the set $\{1, 2, \ldots, 99\}$, then some pair of them adds up to 100.

71. Ten children go on an easter egg hunt and find a total of 44 eggs. Prove that some two children found the same number of eggs.

72. Among any 26 integers between 2 and 100 (inclusive), there exist elements a and b with $\gcd(a, b) > 1$.

73. Consider an 8×8 chessboard with two diagonally opposite corners removed. Prove that it is impossible to cover all remaining 62 squares with just 31 dominoes. (Here, we understand that a "domino" is a rectangle that covers exactly two adjacent squares.)