Bridge to Higher Math
D Term 2011
W. J. Martin
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MA197X Problem Set 5

Instructions: Please review the rules on the presentation of assignments in the course. Then complete the following fourteen problems and submit the solutions, inside your portfolio folder, by Wednesday, April 20th.

For each of the following problems, first state the problem precisely in English and then give a proper proof of the statement using English sentences. Be sure to include the correct problem numbers for recording purposes.

37. FALSE: For all sets X and Y and any function $f : X \to Y$, for all $A, B \subseteq X$, $f(A \cap B) = f(A) \cap f(B)$.
38. FALSE: For all sets X and Y and any function $f : X \to Y$, for all $A, B \subseteq X$, $f(A - B) = f(A) - f(B)$.
39. FALSE: For all sets X and Y and any function $f : X \to Y$, for all $A \subseteq X$, $f^{-1}(f(A)) = A$.
40. FALSE: For all sets X and Y and any function $f : X \to Y$, for all $C \subseteq Y$, $f(f^{-1}(C)) = C$.
41. For all sets X and Y and any function $f : X \to Y$, for all $A, B \subseteq X$, $f(A \cup B) = f(A) \cup f(B)$.
42. For all sets X and Y and any function $f : X \to Y$, for all $C, D \subseteq Y$, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
43. For all sets X and Y and any function $f : X \to Y$, for all $C, D \subseteq Y$, $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.
44. A function $f : \mathbb{R} \to \mathbb{R}$ is *monotonic* if, for all $x, y \in \mathbb{R}$, $x \leq y$ implies $f(x) \leq f(y)$.
    (a) Prove that the composition of two monotonic functions is monotonic.
    (b) FALSE: For all $f, g : \mathbb{R} \to \mathbb{R}$, if $g \circ f$ is monotonic, then at least one of f and g is a monotonic function.
45. A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous* if, for all $a \in \mathbb{R}$, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.
    (a) Prove that the composition of two continuous functions is continuous.
    (b) FALSE: For all $f, g : \mathbb{R} \to \mathbb{R}$, if $g \circ f$ is continuous, then at least one of f and g is a continuous function. *[CHALLENGE: How badly behaved can you make your f and g?]*

46. If f and g are real-valued functions on a set X (i.e., $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$), then $f + g$ is the function $f + g : X \to \mathbb{R}$ defined by the rule $(\forall x \in X)((f + g)(x) = f(x) + g(x))$. Prove: if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions, then $f + g$ is also continuous.

47. Let P be the set of all polynomial functions in one variable:

$$\mathsf{P} = \{f : \mathbb{R} \to \mathbb{R} \mid (\exists n \geq 0)(\exists a_0, a_1, \ldots, a_n)(\forall x \in \mathbb{R})\,(f(x) = a_n x^n + \cdots + a_1 x + a_0)\}\,.$$

Consider the function $D : \mathsf{P} \to \mathsf{P}$ via $D : f \mapsto \frac{df}{dx}$ (simply the first derivative of f).
(a) Prove that D is onto.
(b) Now restrict the domain of D to the subset P_0 of P containing just those polynomial functions with $a_0 = 0$ (i.e., polynomials passing through the origin). Prove that $D : \mathsf{P}_0 \to \mathsf{P}$ is one-to-one and onto.

48. An *infinite sequence* is a function from $\mathbb{N}$ to $\mathbb{R}$. If $a : \mathbb{N} \to \mathbb{R}$, it is customary to write a_n in place of $a(n)$. We say the sequence $a : \mathbb{N} \to \mathbb{R}$ (or $\{a_n\}_{n=1}^{\infty}$) *converges* to limit $L \in \mathbb{R}$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|a_n - L| < \epsilon$. Prove the following theorem:

> If the infinite sequence $\{a_n\}_{n=1}^{\infty}$ converges to limit L and if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then the sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b = f \circ a$ (i.e., for all n, $b_n = f(a_n)$) also converges to a limit in $\mathbb{R}$, namely .... (Finish the statement of the theorem yourself.)

49. True or False? *If a and b are infinite sequences of real numbers, then their composition $g \circ f$ is also an infinite sequence.* (Clearly justify you answer.)

50. A function $f : \mathbb{R} \to \mathbb{R}$ is a *contraction map* with contraction factor c if $0 \leq c < 1$ and, for all $x \in \mathbb{R}$, $|f(x)| \leq c|x|$. Let a_1 be any real number and let $f : \mathbb{R} \to \mathbb{R}$ be a contraction map with contraction factor c. for $n \in \mathbb{N}$ define inductively $a_{n+1} = f(a_n)$. Prove that the resulting sequence $\{a_n\}_{n=1}^{\infty}$ converges to limit $L = 0$. *[NOTE: You may use, without proof, the fact that the geometric sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b_n = cr^n$ converges to zero for any real number c and any ratio $r \in (0, 1)$.]*