Bridge to Higher Math
D Term 2011
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April 1, 2011

MA197X Problem Set 3

Instructions: Please review the rules on the presentation of assignments in the course. Then complete the following ten problems and submit the solutions, inside your portfolio folder, by Thursday, April 7th.

For each of the following problems, first state the problem precisely in English and then give a proper proof of the statement using English sentences. Be sure to include the correct problem numbers for recording purposes.

17. **(a)** Prove: For all sets A and B and all relations $\boldsymbol{r}$, $\boldsymbol{s}$ from A to B, we have $\mathrm{Dom}\,(\boldsymbol{r} \cup \boldsymbol{s}) = \mathrm{Dom}\,(\boldsymbol{r}) \cup \mathrm{Dom}\,(\boldsymbol{s})$. (Then show that, without further proof, it follows that $\mathrm{Im}\,(\boldsymbol{r} \cup \boldsymbol{s}) = \mathrm{Im}\,(\boldsymbol{r}) \cup \mathrm{Im}\,(\boldsymbol{s})$.)

    (b) Show that the following proposition is false: For all sets A and B and all relations $\boldsymbol{r}$, $\boldsymbol{s}$ from A to B, we have $\mathrm{Dom}\,(\boldsymbol{r} \cap \boldsymbol{s}) = \mathrm{Dom}\,(\boldsymbol{r}) \cap \mathrm{Dom}\,(\boldsymbol{s})$.

18. Prove: If $\boldsymbol{r}$ is a relation on set A with $\mathrm{Dom}\,(\boldsymbol{r}) = A$ and $\boldsymbol{r}$ is both symmetric and transitive, then $\boldsymbol{r}$ is reflexive.

19. Prove: If A is any set and $\boldsymbol{r}$ is a relation on A, then $\boldsymbol{r}$ is both symmetric and antisymmetric if and only if $\boldsymbol{r} \subseteq \mathrm{id}_A := \{(a,a) \ : \ a \in A\}$.

20. Supppose A is a non-empty set and consider the relation $\boldsymbol{r}$ defined on $\mathcal{P}(A)$ by

    $$A\boldsymbol{r}B \ \leftrightarrow \ A \cap B = \emptyset.$$

    In parts (a)-(e), decide whether the given statement is TRUE or FALSE. If it is true, provide a proof; if it is false, provide a simple counterexample.

    (a) $\boldsymbol{r}$ is reflexive

    (b) $\boldsymbol{r}$ is irreflexive

    (c) $\boldsymbol{r}$ is symmetric

    (d) $\boldsymbol{r}$ is antisymmetric

    (e) $\boldsymbol{r}$ is transitive

21. Let A, B and C be sets. Let $\boldsymbol{r}$ be a relation from A to B and let $\boldsymbol{s}$ be a relation from B to C. For these objects, define

$$\boldsymbol{s} \circ \boldsymbol{r} = \{(a, c) \in A \times C \mid (\exists b \in B)\,(a\boldsymbol{r}b \wedge b\boldsymbol{s}c)\}\,.$$

Prove: For any A, B, C and any $\boldsymbol{r} \subseteq A \times B$ and $\boldsymbol{s} \subseteq B \times C$, $\mathrm{Dom}\,(\boldsymbol{s} \circ \boldsymbol{r}) \subseteq \mathrm{Dom}\,(\boldsymbol{r})$ and $\mathrm{Im}\,(\boldsymbol{s} \circ \boldsymbol{r}) \subseteq \mathrm{Im}\,(\boldsymbol{s})$.

22. With notation as in Problem 21, prove: if $\mathrm{Im}\,(\boldsymbol{r}) = \mathrm{Dom}\,(\boldsymbol{s})$, then $\mathrm{Dom}\,(\boldsymbol{s} \circ \boldsymbol{r}) = \mathrm{Dom}\,(\boldsymbol{r})$ and $\mathrm{Im}\,(\boldsymbol{s} \circ \boldsymbol{r}) = \mathrm{Im}\,(\boldsymbol{s})$. *[NOTE: If you have previously solved Problem 21, then you may use its result in your solution.]*

23. Let A be a non-empty set and let $\boldsymbol{r}$ and $\boldsymbol{s}$ be relations on A. For each of the following propositions, decide whether the statement is true or false. If it is true, prove it; if the statement is false, give a simple counterexample.

    (a) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are reflexive, then $\boldsymbol{s} \circ \boldsymbol{r}$ is reflexive.

    (b) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are irreflexive, then $\boldsymbol{s} \circ \boldsymbol{r}$ is irreflexive.

    (c) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric, then $\boldsymbol{s} \circ \boldsymbol{r}$ is symmetric.

    (d) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are antisymmetric, then $\boldsymbol{s} \circ \boldsymbol{r}$ is antisymmetric.

    (e) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are transitive, then $\boldsymbol{s} \circ \boldsymbol{r}$ is transitive.

24. In number theory, we make extensive use of the "exactly divides" relation. The relation $\| \subseteq \mathbb{Z} \times \mathbb{Z}$ is defined as follows: for a prime p, and integer n and a positive integer k,

$$p^k \| n \;\leftrightarrow\; \left[(p^k | n) \;\wedge\; (\forall \ell \in \mathbb{Z})\left(\ell > k \rightarrow\; p^\ell \nmid n\right)\right];$$

in all other cases, $m \| n$ is false.

    (a) Find $\mathrm{Dom}\,(\|)$. Explain briefly.

    (b) Find $\mathrm{Im}\,(\|)$. Explain briefly.

    (c) Show that, for any prime p and any $k \geq 1$, the set $\left\{n \in \mathbb{Z} \;:\; p^k \| n\right\}$ is infinite.

    (d) For $n \in \mathbb{Z}$, *arbitrary but fixed*, what can you conclude about the size of the set $\{m \in \mathbb{Z} \;:\; m \| n\}$? Justify.

25. Let m and n be positive integers. Let $\boldsymbol{r}$ be the relation "congruence modulo m" on $\mathbb{Z}$ and let $\boldsymbol{s}$ be the relation "congruence modulo n" on $\mathbb{Z}$ (see p44 for the definition). Prove: if $n|m$, then $\boldsymbol{r} \subseteq \boldsymbol{s}$.

26. Prove: For any positive integer n and for all integers a, b, c, d, if $a \equiv b \bmod n$ and $c \equiv d \bmod n$, then

$$a + c \equiv b + d \bmod n \qquad \text{and} \qquad ac \equiv bd \bmod n.$$