Proofs in Contemporary Math
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MA196X Problem Set 6

Instructions: Please first read the rules on the presentation of assignments in the course. Then complete as many of these as you can by Wednesday, May 6th, which will be the final deadline for submission of your portfolio containing your forty best solutions.

Note: Always identify each problem by its problem number and re-state the problem precisely before giving its solution.

To some, mathematics is the study of functions. Each area of pure mathematics focuses on certain structures (e.g., vector spaces) on sets and functions from one such set to another which preserve this structure (e.g., linear transformations). In any such study, the first things we prove are (i) the identity function from a set to itself preserves structure, and (ii) the composition of structure-preserving functions is always structure-preserving. Since (i) is usually very easy to prove, we will not include problems of that sort here but the first three problems below are special cases of property (ii).

64. Prove that the composition of homomorphisms is a homomorphism.

    In algebra, we consider structures consisting of sets G endowed with functions

    $$\star : G \times G \to G$$

    (called *operations*) satisfying various properties such as associativity, commutativity, and existence of an identity. If $(G, \star)$ and $(H, \bullet)$ are two such algebraic structures, a function

    $$f : (G, \star) \to (H, \bullet)$$

    (i.e., f is a function from the set G to the set H) is a *homomorphism* if, for all $a, b \in G$, $f(a \star b) = f(a) \bullet f(b)$. Prove: if $f : (G, \star) \to (H, \bullet)$ and $g : (H, \bullet) \to (K, \cdot)$ are homomorphisms, then $g \circ f : (G, \star) \to (K, \cdot)$ is also a homomorphism.

65. Prove that the composition of order-preserving maps is an order-preserving map.

    Recall that a *partially ordered set* (or "poset") is an ordered pair $(X, \preceq)$ where X is a set and $\preceq$ is a partial order relation[1] on X. If $(X, \preceq)$ and $(Y, \unlhd)$ are posets, a function $f : (X, \preceq) \to (Y, \unlhd)$ is an *order-preserving map* if, for all $a, b \in X$, $a \preceq b$ implies $f(a) \unlhd f(b)$. Prove: for any posets $(X, \preceq)$, $(Y, \unlhd)$ and $(Z, \leq)$ and any order-preserving maps $f : (X, \preceq) \to (Y, \unlhd)$ and $g : (Y, \unlhd) \to (Z, \leq)$, the composition $g \circ f$ is an order-preserving map from $(X, \preceq)$ to $(Z, \leq)$.

[1] That is, a relation $\preceq \subseteq X \times X$ which is reflexive, antisymmetric and transitive.

66. Prove that the composition of continuous functions is continuous.

Suppose D is a distance function[2] on set X and d is a distance function on set Y. A function

$$f : X \to Y$$

is said to be *continuous* (or, "a map") with respect to D and d if, for every $x \in X$ and for every $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $y \in X$, if $D(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$. To emphasize that continuity depends not only on X and Y but also on their respective distance functions D and d, we write

$$f : (X, D) \to (Y, d) \qquad \text{is continuous.}$$

Now for your problem.

Suppose D is a distance function on set X and d is a distance function on set Y and ∂ is a distance function on set Z. If $f : (X, D) \to (Y, d)$ and $g : (Y, d) \to (Z, \partial)$ are any continuous functions, then their composition $g \circ f$ is a continuous function from (X, D) to (Z, ∂).

67. Consider two distance functions on the real plane $\mathbb{R}^2$:

$$d_2(x, y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}, \qquad d_1(x, y) = |y_1 - x_1| + |y_2 - x_2|.$$

Prove that the concept of continuity is the same for these two metrics[3]. That is, if ∂ is any distance function on any set Y and $f : (\mathbb{R}^2, d_2) \to (Y, \partial)$ is any continuous function, then $f : (\mathbb{R}^2, d_1) \to (Y, \partial)$ is also continuous. And vice versa.

68. Let X and Y be sets and let $f : X \to Y$. For any relation $\sim$ on Y, define a relation $\approx$ on X via $a \approx b$ if $f(a) \sim f(b)$. Prove: if $\sim$ is an equivalence relation, then so also is $\approx$.

69. "HOW TO MAKE ANY FUNCTION ONE-TO-ONE": Let $f : X \to Y$ be any function. Consider the relation $\sim$ on X given by $a \sim b$ whenever $f(a) = f(b)$.
(a) Prove that $\sim$ is an equivalence relation on X;

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[2] A *distance function* (or metric) on a set X is a function

$$D : X \times X \to \mathbb{R}$$

satisfying four conditions: *(i)* $D(x, y) = D(y, x)$ for all $x, y \in X$; *(ii)* $D(x, y) \geq 0$ for all $x, y \in X$; *(iii)* for all $x, y \in X$, $D(x, y) = 0$ iff $x = y$; *(iv)* $D(x, z) \leq D(x, y) + D(y, z)$ for all $x, y, z \in X$. An ordered pair (X, D) where X is a set and D is a distance function on X is called a *metric space* and these are fundamental structures in topology.

[3] The first is of course the usual Euclidean metric while the second is called the "taxicab metric" or "Manhattan metric".

(b) Let $X/\sim\, = \{[a] : a \in X\}$ denote the set of equivalence classes of X under $\sim$. Prove that $\tilde{f} : X/\sim\, \to Y$ defined by

$$\tilde{f}(S) = y \quad \text{if} \quad (\exists a \in S)\,(f(a) = y)$$

is both well-defined and one-to-one.

70. Let X and Y be sets and let $f : X \to Y$. For any relation $\leq$ on Y, define a relation $\preceq$ on X via $a \preceq b$ if either $a = b$ or if $f(a) < f(b)$ (i.e., $f(a) \leq f(b)$ but $f(a) \neq f(b)$). Prove: if $\leq$ is a partial order relation, then so also is $\preceq$.

71. A function f from a set X to itself is called an *involution* when $f \circ f = I_X$, the identity function on X. Prove from first principles:
(a) for any set X and any involution $f : X \to X$, f is one-to-one.
(b) for any set X and any involution $f : X \to X$, f is onto.

72. For sets X and Y, let Y^X denote the set of *all* possible functions $f : X \to Y$. (So, e.g., we use $\mathbb{R}^3$ as shorthand for the set of all functions from $\{1, 2, 3\}$ to the real numbers.) For fixed $a \in X$, consider the *evaluation map* $\varepsilon_a : Y^X \to Y$ defined by $\varepsilon_a(f) = f(a)$. Prove: For any sets X and Y and any $a \in X$, (i) ε_a is always onto, and (ii) if ε_a is one-to-one, then either $|X| = 1$ or $|Y| = 1$.

73. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(a, b) = \int_a^b x^2\, dx$. Prove that f is onto, but not one-to-one. What about $g : \mathbb{R} \to \mathbb{R}$ given by $g(b) = f(1, b)$?

74. Disprove: For all functions $f : [0, \infty) \to \mathbb{R}$, if the function $F(x) = \int_0^x f(t)\, dt$ is continuous, then f itself is continuous. (Assume here that f is *integrable* on $[0, \infty)$; that is, assume that F is a function.)

75. Let $f : \mathbb{R} \to \mathbb{R}$ be any continuous function. We say f is an *even function* (resp. *odd function*) if, for all x, $f(-x) = f(x)$ (resp., $f(-x) = -f(x)$). Prove: if f is an even function, then $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \int_0^x f(t)\, dt$ is an odd function (and if f is an odd function, then F is an even function). You may use elementary properties of definite integrals in your proof.