Proofs in Contemporary Math
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MA196X Problem Set 5

Instructions: Please first read the rules on the presentation of assignments in the course. Then complete as many of these as you can by Friday, April 24th. After that, I will still accept problems until the sample solutions have been distributed.

Note: All statements here are to be proven using the Principle of Mathematical Induction. Always identify each problem by its problem number and re-state the problem precisely before giving its solution.

46. For all positive integers n,
$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

47. For all positive integers n
$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$
[HINT: You may want to use a proof by contradiction in your induction step.]

48. For all positive integers n
$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

49. For every positive integer n, the integer $n^3 + 5n$ is divisible by six.

50. For every positive integer n,
$$1 + \frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n} = \frac{3}{2}\left(1 - \frac{1}{3^{n+1}}\right).$$

51. For every integer $n \geq 4$, $n! > 2^n$. *[NOTE: This requires a straightforward variation on the Principle of Mathematical Induction. Please state it — without proof — in your "Recall" section.]*

52. For every positive integer n
$$2\left(1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}\right) > 2\sqrt{n+1}.$$

53. For every positive integer n

$$2\left(1+\frac{1}{8}+\cdots+\frac{1}{n^3}\right)<3-\frac{1}{n^2}.$$

In the remaining problems (except the last two), you need to find the theorem before you search for its proof. Using experimentation with small values of n, first make a conjecture regarding the outcome for general positive integers n and then prove your conjecture using induction. (NOTE: The experimentation should be done on scrap paper and is not part of your formal solution.)

54. Simplify $1+2+4+\cdots+2^n$.

55. Find a simple expression for the sum

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}.$$

56. For a real number a which is not an integer, simplify

$$\frac{1}{a\cdot(a+1)}+\frac{1}{(a+1)\cdot(a+2)}+\cdots+\frac{1}{(a+n-1)\cdot(a+n)}.$$

57. What is the largest integer that divides $n^3+(n+1)^3+(n+2)^3$ for every positive integer n?

58. If n lines are drawn in the plane so that no two are parallel and no three share a common point, into how many regions is the plane divided by these lines?

59. Let us call a positive integer n "posh" if it is possible to dissect a square into n smaller squares. For example, $n=6$ and $n=7$ are posh while $n=1$ and $n=2$ are not. Which natural numbers are posh?

60. The Fibonacci numbers[1] are defined by $F_1=F_2=1$ and $F_n=F_{n-1}+F_{n-2}$ for all $n\geq 3$. Find the value of $F_{n-1}F_{n+1}-F_n^2$ for $n\geq 2$.

61. Find a relation for $F_n^2+F_{n+1}^2$ which holds for all positive integers n.

62. If $m|n$, then F_m divides F_n. *[HINT: Use induction on $k=n/m$.]*

63. Every positive integer can be expressed as a sum of distinct Fibonacci numbers.

[1]These numbers were introduced in Year 1202 by Leonardo of Pisa, the son of Bonacci.