Proofs in Contemporary Math
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MA196X Problem Set 4

Instructions: Please first read the rules on the presentation of assignments in the course. Then complete as many of these as you can by Friday, April 24th. After that, I will still accept problems until the sample solutions have been distributed.

Note: Always identify each problem by its problem number and re-state the problem precisely before giving its solution.

25. For all sets A, B, C and D, and any functions f, g and h, if $f : A \to B$, $g : B \to C$ and $h : C \to D$, then
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

26. For all sets A and B and for any function $f : A \to B$,
$$I_B \circ f = f \qquad \text{and} \qquad f \circ I_A = f.$$

27. For all sets A, B and C and any functions $f : A \to B$ and $g : B \to C$
(a) if f and g are both injective, then $g \circ f$ is also injective.
(b) if $g \circ f$ is an injection, then f is also an injection.

    Finally, in part **(c)**, give a simple example of sets A, B, C and functions f, g as above where $g \circ f$ is injective, but g is not.

28. For all sets A, B and C and any functions $f : A \to B$ and $g : B \to C$
(a) if f and g are both surjective, then $g \circ f$ is also surjective.
(b) if $g \circ f$ is an surjection, then g is also a surjection.

    Finally, in part **(c)**, give a simple example of sets A, B, C and functions f, g as above where $g \circ f$ is surjective, but f is not.

29. Prove that the following proposition is FALSE: *For all sets A and B and all functions $f : A \to B$ and $g : B \to A$, if $g \circ f = I_A$, then $f \circ g = I_B$.*

30. For all sets A and B and all $f : A \to B$ if f is both one-to-one and onto, then $(f^{-1})^{-1} = f$.

31. For all sets A and B and all $f : A \to B$ and $g : B \to A$, if f and g are both bijections and $g \circ f = I_A$ then $g = f^{-1}$.

32. Throughout parts (a)–(c), assume that X and Y are sets and $f : X \to Y$.
(a) For all $A, B \subseteq X$, $f(A \cup B) = f(A) \cup f(B)$.
(b) For all $A, B \subseteq X$, $f(A \cap B) \subseteq f(A) \cap f(B)$.
(c) For all $A, B \subseteq X$, $f(A - B) \subseteq f(A) - f(B)$.

33. Throughout parts (a)–(c), assume that X and Y are sets and $f : X \to Y$.
(a) For all $C, D \subseteq Y$, $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.
(b) For all $C, D \subseteq Y$, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.
(c) For all $C, D \subseteq Y$, $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.

34. For any sets X and Y and any $f : X \to Y$
(a) for any $A, B \subseteq X$, if $A \subseteq B$, then $f(A) \subseteq f(B)$;
(b) for any $A \subseteq X$, $A \subseteq f^{-1}(f(A))$.

35. For any sets X and Y and any $f : X \to Y$
(a) for any $C, D \subseteq Y$, if $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$;
(b) for any $C \subseteq Y$, $f(f^{-1}(C)) \subseteq C$.

36. For any sets X, Y and Z and any $f : X \to Y$ and $g : Y \to Z$
(a) for all $A \subseteq X$, $(g \circ f)(A) = g(f(A))$;
(b) for all $C \subseteq Z$, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$.

37. Let X and Y be sets and let $f : X \to Y$. Then f is one-to-one if and only if for all $A, B \subseteq X$, $f(A \cap B) = f(A) \cap f(B)$.

38. Let X and Y be sets and let $f : X \to Y$. Then f is one-to-one if and only if for all $A, B \subseteq X$, $f(A - B) = f(A) - f(B)$.

39. Let X and Y be sets and let $f : X \to Y$. Then f is one-to-one if and only if for all $A \subseteq X$, $f^{-1}(f(A)) = A$.

40. Let X and Y be sets and let $f : X \to Y$. If $f(f^{-1}(Y)) = Y$ then f is onto.

41. Let X and Y be sets and let $f : X \to Y$. If f is onto then, for all $C \subseteq Y$, $f(f^{-1}(C)) = C$.

42. Let A, B, C and D be sets and let $f : A \to C$ and $g : B \to D$. Define a function $H : A \times B \to C \times D$ by
$$H(a, b) = (f(a), g(b))$$
for all $(a, b) \in A \times B$.
(a) Prove: if f and g are both one-to-one, then H is also one-to-one.
(b) Disprove: if f and g are both onto, then H is also onto.

43. Let n be a positive integer and consider $f : \mathbb{R}^n \to \mathbb{R}$ given by
$$f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n.$$
Prove that f is onto, but that f is one-to-one if and only if $n = 1$.

44. Let A be a set and let $\sim$ be an equivalence relation on A. Prove that the rule $a \mapsto [a]$, which sends each $a \in A$ to the equivalence class containing it, is a function from A to $A/\sim$ and in fact a surjection. (In many parts of mathematics, this is called the "natural map" and denoted ν.)

45. Let m and n be positive integers and, for any integer a, let $[a]_m$ (resp. $[a]_n$) denote the equivalence class containing a under congruence modulo m (resp. n). Define $\mathbb{Z}_m = \mathbb{Z}/\equiv_m$ and $\mathbb{Z}_n = \mathbb{Z}/\equiv_n$ to be the corresponding collections of equivalence classes. Prove that the rule f given by

$$f : [a]_m \mapsto [a]_n$$

is a function from $\mathbb{Z}_m$ to $\mathbb{Z}_n$ if and only if $n|m$.