Proofs in Contemporary Math
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MA196X Problem Set 3

Instructions: Please first read the rules on the presentation of assignments in the course. Then complete as many of these as you can by Friday, April 10th. After that, I will still accept problems until the sample solutions have been distributed.

Note: Always identify each problem by its problem number and re-state the problem precisely before giving its solution.

13. **(a)** Prove: For all sets A and B and all relations $\boldsymbol{r}$, $\boldsymbol{s}$ from A to B, we have $\mathrm{Dom}\,(\boldsymbol{r} \cup \boldsymbol{s}) = \mathrm{Dom}\,(\boldsymbol{r}) \cup \mathrm{Dom}\,(\boldsymbol{s})$. (Then show that, without further proof, it follows that $\mathrm{Im}\,(\boldsymbol{r} \cup \boldsymbol{s}) = \mathrm{Im}\,(\boldsymbol{r}) \cup \mathrm{Im}\,(\boldsymbol{s})$.)

    (b) Show that the following proposition is false: For all sets A and B and all relations $\boldsymbol{r}$, $\boldsymbol{s}$ from A to B, we have $\mathrm{Dom}\,(\boldsymbol{r} \cap \boldsymbol{s}) = \mathrm{Dom}\,(\boldsymbol{r}) \cap \mathrm{Dom}\,(\boldsymbol{s})$.

14. Prove: If $\boldsymbol{r}$ is a relation on set A with $\mathrm{Dom}\,(\boldsymbol{r}) = A$ and $\boldsymbol{r}$ is both symmetric and transitive, then $\boldsymbol{r}$ is reflexive.

15. Prove: If A is any set and $\boldsymbol{r}$ is a relation on A, then $\boldsymbol{r}$ is both symmetric and antisymmetric if and only if $\boldsymbol{r} \subseteq \mathrm{id}_A := \{(a,a) \ : \ a \in A\}$.

16. Supppose A is a non-empty set and consider the relation $\boldsymbol{r}$ defined on $\mathcal{P}(A)$ by

    $$A\boldsymbol{r}B \ \leftrightarrow \ A \cap B = \emptyset.$$

    In parts (a)-(e), decide whether the given statement is TRUE or FALSE. If it is true, provide a proof; if it is false, provide a simple counterexample.

    (a) $\boldsymbol{r}$ is reflexive

    (b) $\boldsymbol{r}$ is irreflexive

    (c) $\boldsymbol{r}$ is symmetric

    (d) $\boldsymbol{r}$ is antisymmetric

    (e) $\boldsymbol{r}$ is transitive

17. Let A, B and C be sets. Let $\boldsymbol{r}$ be a relation from A to B and let $\boldsymbol{s}$ be a relation from B to C. For these objects, define

    $$\boldsymbol{s} \circ \boldsymbol{r} = \{(a,c) \in A \times C \mid (\exists b \in B)\,(a\boldsymbol{r}b \wedge b\boldsymbol{s}c)\}\,.$$

    Prove: For any A, B, C and any $\boldsymbol{r} \subseteq A \times B$ and $\boldsymbol{s} \subseteq B \times C$, $\mathrm{Dom}\,(\boldsymbol{s} \circ \boldsymbol{r}) \subseteq \mathrm{Dom}\,(\boldsymbol{r})$ and $\mathrm{Im}\,(\boldsymbol{s} \circ \boldsymbol{r}) \subseteq \mathrm{Im}\,(\boldsymbol{s})$.

18. With notation as in the previous problem, prove: if $B = C$ and $\boldsymbol{s} = \mathrm{id}_B$, then $\boldsymbol{s} \circ \boldsymbol{r} = \boldsymbol{r}$.

19. With notation as in Problem 17, prove: if $\mathrm{Im}\,(\boldsymbol{r}) = \mathrm{Dom}\,(\boldsymbol{s})$, then $\mathrm{Dom}\,(\boldsymbol{s} \circ \boldsymbol{r}) = \mathrm{Dom}\,(\boldsymbol{r})$ and $\mathrm{Im}\,(\boldsymbol{s} \circ \boldsymbol{r}) = \mathrm{Im}\,(\boldsymbol{s})$. *[NOTE: If you have previously solved Problem 17, then you may use its result in your solution.]*

20. Let A be a non-empty set and let $\boldsymbol{r}$ and $\boldsymbol{s}$ be relations on A. For each of the following propositions, decide whether the statement is true or false. If it is true, prove it; if the statement is false, give a simple counterexample.

    (a) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are reflexive, then $\boldsymbol{s} \circ \boldsymbol{r}$ is reflexive.

    (b) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are irreflexive, then $\boldsymbol{s} \circ \boldsymbol{r}$ is irreflexive.

    (c) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric, then $\boldsymbol{s} \circ \boldsymbol{r}$ is symmetric.

    (d) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are antisymmetric, then $\boldsymbol{s} \circ \boldsymbol{r}$ is antisymmetric.

    (e) If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are transitive, then $\boldsymbol{s} \circ \boldsymbol{r}$ is transitive.

21. In number theory, we make extensive use of the "exactly divides" relation. The relation $\| \subseteq \mathbb{Z} \times \mathbb{Z}$ is defined as follows: for a prime p, and integer n and a positive integer k,

$$p^k \| n \;\leftrightarrow\; \left[(p^k | n) \;\wedge\; (\forall \ell \in \mathbb{Z}) \left(\ell > k \rightarrow\; p^\ell \nmid n \right) \right];$$

    in all other cases, $m \| n$ is false.

    (a) Find $\mathrm{Dom}\,(\|)$. Explain briefly.

    (b) Find $\mathrm{Im}\,(\|)$. Explain briefly.

    (c) Show that, for any prime p and any $k \geq 1$, the set $\left\{ n \in \mathbb{Z} \;:\; p^k \| n \right\}$ is infinite.

    (d) For $n \in \mathbb{Z}$, *arbitrary but fixed*, what can you conclude about the size of the set $\{ m \in \mathbb{Z} \;:\; m \| n \}$? Justify.

22. Let m and n be positive integers. Let $\boldsymbol{r}$ be the relation "congruence modulo m" on $\mathbb{Z}$ and let $\boldsymbol{s}$ be the relation "congruence modulo n" on $\mathbb{Z}$ (see p44 for the definition). Prove: if $n | m$, then $\boldsymbol{r} \subseteq \boldsymbol{s}$.

23. Prove: For any positive integer n and for all integers a, b, c, d, if $a \equiv b \bmod n$ and $c \equiv d \bmod n$, then

$$a + c \equiv b + d \bmod n \qquad \text{and} \qquad ac \equiv bd \bmod n.$$

24. Let $(A, \preceq)$ be a finite poset (i.e., A is a finite set and $\preceq$ is a partial order relation on A). Prove that there exists a linear extension for $\preceq$: there exists a total order relation $\preceq_*$, extending $\preceq$ (i.e., $\preceq \subseteq \preceq_* \subseteq A \times A$). (See p61 for the definition.)