Proofs in Contemporary Math
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March 20, 2009

MA196X Problem Set 1

Instructions: Please first read the rules on the presentation of assignments in the course. Then complete as many of these as you can by Tuesday, March 24th. After that, I will still accept problems until the sample solutions have been distributed.

For each of the following problems, first state the problem precisely and then give a proper proof of the statement using English sentences.

1. The following hold for any integers:
   (a) If $a|b$ and $b|a$, then $b = \pm a$;
   (b) If a and b are positive and $a|b$, then $a \leq b$;
   (c) $a|a$;
   (d) If a and b are positive and $a|b$ and $b|a$, then $b = a$;
   (e) If $a|b$ and $b|c$, then $a|c$.
2. The following hold for any integers:
   (a) If $a|b$ then $a|bx$ for any integer x;
   (b) If $a|b$ and $a|c$, then $a|(bx + cy)$ for any integers x and y;
   (c) If $a|b$ and $c|d$, then $ac|bd$.
3. The following are all false:
   (a) For all integers a, b, c, if $a|bc$ then either $a|b$ or $a|c$;
   (b) For all integers a, b, c, d, if $a|b$ and $c|d$ then $(a + c)|(b + d)$;
   (c) For all integers a, b, c, if $a \nmid b$ and $a \nmid c$ then $a \nmid bc$;
   (d) For all integers a, b, c, if $a \nmid b$ and $b \nmid c$ then $a \nmid c$.
4. If n is an odd integer, then $8|(n^2 - 1)$.
5. If p is prime and $p|ab$, then $p|a$ or $p|b$.

   [HINT: For this more challenging problem, you may use the following theorem without proof: If integers c and d are relatively prime, then there exist integers x and y such that $cx + dy = 1$. *Two integers are* **relatively prime** *if they have no common divisor larger than one, as is true for $c = 15$ and $d = 28$ for example. For this same example, the values $x = -13$, $y = 7$ give $15x + 28y = 1$ as desired.]*

6. Consider the following two conjectures:

Conjecture A: For every positive integer n, there exists a prime number between n and n^2.

Conjecture B: For every positive integer n, there exists a prime number between n and $2n$.

While we currently do not know if either of these is true, one implies the other. Figure out which implies which and prove this implication.