Proofs in Contemporary Math
D Term 2009
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Sample Solution – Proofs by Induction

Proposition: For every positive integer n,

$$\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}.$$

Proof *(by induction)***:** Let P denote the set of positive integers for which the given statement is true:

$$P=\left\{n\in\mathbb{N}\;\middle|\;\sum_{k=1}^{n}\frac{1}{k(k+1)(k+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\right\}.$$

Base Case: We prove the statement for $n=1$. Since

$$\frac{1}{1\cdot 2\cdot 3}=\frac{1}{6}=\frac{1}{4}-\frac{1}{12}=\frac{1}{4}-\frac{1}{2(1+1)(1+2)},$$

we have shown $1\in P$.

IHOP: We let $n\geq 1$ be arbitrarily chosen and assume the statement is true for n; i.e., suppose $n\in P$.

Induction Step: We aim to show that the statement is true for $n+1$. We use the induction hypothesis to write

$$\begin{aligned}
\frac{1}{1\cdot 2\cdot 3} &+ \frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)} = \\
\left(\frac{1}{1\cdot 2\cdot 3}\right. &+ \left.\frac{1}{2\cdot 3\cdot 4}+\cdots+\frac{1}{n(n+1)(n+2)}\right)+\frac{1}{(n+1)(n+2)(n+3)} \\
&= \left(\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\right)+\frac{1}{(n+1)(n+2)(n+3)} \\
&= \frac{1}{4}-\frac{n+3}{2(n+1)(n+2)(n+3)}+\frac{2}{2(n+1)(n+2)(n+3)} \\
&= \frac{1}{4}+\frac{2-n-3}{2(n+1)(n+2)(n+3)} \\
&= \frac{1}{4}+\frac{-n-1}{2(n+1)(n+2)(n+3)} \\
&= \frac{1}{4}-\frac{n+1}{2(n+1)(n+2)(n+3)} \\
&= \frac{1}{4}-\frac{1}{2(n+2)(n+3)}.
\end{aligned}$$

This sequence of equations shows

$$\sum_{k=1}^{n+1}=\frac{1}{k(k+1)(k+2)}=\frac{1}{4}-\frac{1}{2(n+2)(n+3)}$$

which proves that $n+1\in P$.

By the Principle of Mathematical Induction, we conclude that $P=\mathbb{N}$. Q.E.D.