Proofs in Contemporary Math
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MA196X Practice Worksheet

Instructions: These are **not** homework problems. This handout consists of a large number of simple mathematical statements. Some are true and some are false. We will use these problems for classroom presentation, problem-solving in discussion groups (i.e., when – as university students – you want to work on problems together outside of class), and in office hours. For each statement, the first step is to experiment and/or analyze to decide if the statement is true or not. If it is true, then write a proper proof. If it is false, give its negation and provide a counterexample.

Note: Not all problems are properly stated. In mathematics, some quantifiers are implied and must be accounted for by the one who solves the problem.

The first few problems deal with Cartesian products and power sets.

1. For any sets A and B, $A \times B = B \times A$.
2. For any sets A, B and C, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
3. For any sets A, B and C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.
4. For any sets A, B and C, $A \times (B - C) = (A \times B) - (A \times C)$.
5. For any sets A and B, $\mathcal{P}\,(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.
6. For any sets A and B, $\mathcal{P}\,(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.
7. For any sets A and B, $\mathcal{P}\,(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.
8. For any sets A and B, $\mathcal{P}\,(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$.

Throughout the next set of problems, suppose $\boldsymbol{r}$ and $\boldsymbol{s}$ are any relations on any set A.

9. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are reflexive, then $\boldsymbol{r} \cap \boldsymbol{s}$ is reflexive.
10. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are reflexive, then $\boldsymbol{r} \cup \boldsymbol{s}$ is reflexive.
11. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are reflexive, then $\boldsymbol{r} - \boldsymbol{s}$ is irreflexive.
12. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are irreflexive, then $\boldsymbol{r} \cap \boldsymbol{s}$ is irreflexive.

13. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are irreflexive, then $\boldsymbol{r} \cup \boldsymbol{s}$ is irreflexive.

14. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are irreflexive, then $\boldsymbol{r} - \boldsymbol{s}$ is irreflexive.

15. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric, then $\boldsymbol{r} \cap \boldsymbol{s}$ is symmetric.

16. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric, then $\boldsymbol{r} \cup \boldsymbol{s}$ is symmetric.

17. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric, then $\boldsymbol{r} - \boldsymbol{s}$ is symmetric.

18. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric, then $\boldsymbol{r} - \boldsymbol{s}$ is antisymmetric.

19. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are antisymmetric, then $\boldsymbol{r} \cap \boldsymbol{s}$ is antisymmetric.

20. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are antisymmetric, then $\boldsymbol{r} \cup \boldsymbol{s}$ is antisymmetric.

21. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are antisymmetric, then $\boldsymbol{r} - \boldsymbol{s}$ is antisymmetric.

22. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are transitive, then $\boldsymbol{r} \cap \boldsymbol{s}$ is transitive.

23. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are transitive, then $\boldsymbol{r} \cup \boldsymbol{s}$ is transitive.

24. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are transitive, then $\boldsymbol{r} - \boldsymbol{s}$ is transitive.

Again, suppose $\boldsymbol{r}$ and $\boldsymbol{s}$ are any relations on a set A. As you can see, these are similar to the ones above.

25. If $\boldsymbol{r} \cap \boldsymbol{s}$ is reflexive, then both $\boldsymbol{r}$ and $\boldsymbol{s}$ are reflexive.

26. If $\boldsymbol{r}$ is reflexive, then $\boldsymbol{r} \cup \boldsymbol{s}$ is reflexive.

27. If $\boldsymbol{r}$ is reflexive and $\boldsymbol{s}$ is irreflexive, then $\boldsymbol{r} - \boldsymbol{s}$ is reflexive.

28. If $\boldsymbol{r}$ is irreflexive, then $\boldsymbol{r} \cap \boldsymbol{s}$ is irreflexive.

29. If $\boldsymbol{r} \cup \boldsymbol{s}$ is irreflexive, then both $\boldsymbol{r}$ and $\boldsymbol{s}$ are irreflexive.

30. If $\boldsymbol{r}$ is irreflexive, then $\boldsymbol{r} - \boldsymbol{s}$ is irreflexive.

31. If $\boldsymbol{r} \cap \boldsymbol{s}$ is symmetric, then both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric.

32. If $\boldsymbol{r} \cup \boldsymbol{s}$ is symmetric, then both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric.

33. If $\boldsymbol{r}$ is symmetric, then $\boldsymbol{r} - \boldsymbol{s}$ is symmetric.

34. If $\boldsymbol{r}$ is antisymmetric, then $\boldsymbol{r} \cap \boldsymbol{s}$ is antisymmetric.

35. If $\boldsymbol{r} \cup \boldsymbol{s}$ is antisymmetric, then both $\boldsymbol{r}$ and $\boldsymbol{s}$ are antisymmetric.

36. If $\boldsymbol{r}$ is antisymmetric, then $\boldsymbol{r} - \boldsymbol{s}$ is antisymmetric.

37. If $\boldsymbol{r} \cup \boldsymbol{s}$ is transitive, then both $\boldsymbol{r}$ and $\boldsymbol{s}$ are transitive.

38. If $\boldsymbol{r} \cap \boldsymbol{s}$ is transitive, then both $\boldsymbol{r}$ and $\boldsymbol{s}$ are transitive.

Throughout, suppose A is a set, $\boldsymbol{r}$ is any relation on A and $\boldsymbol{r}^{-1}$ is the *inverse* relation

$$\boldsymbol{r}^{-1} = \{(b, a) \mid (a, b) \in \boldsymbol{r}\}.$$

39. If $\boldsymbol{r}$ is reflexive, then $\boldsymbol{r}^{-1}$ is reflexive.

40. If $\boldsymbol{r}$ is irreflexive, then $\boldsymbol{r}^{-1}$ is irreflexive.

41. $\boldsymbol{r}^{-1} = \boldsymbol{r}$ if and only if $\boldsymbol{r}$ is symmetric.

42. If $\boldsymbol{r}$ is antisymmetric, then $\boldsymbol{r}^{-1}$ is antisymmetric.

43. If $\boldsymbol{r}$ is transitive, then $\boldsymbol{r}^{-1}$ is transitive.

44. If $\boldsymbol{r}$ is a partial order relation, then $\boldsymbol{r}^{-1}$ is a partial order relation.

45. If $\boldsymbol{r}$ is reflexive and transitive, then $\boldsymbol{r} \cup \boldsymbol{r}^{-1}$ is an equivalence relation.

For the next set of problems let $\boldsymbol{r}$ be a relation on set A and let $\boldsymbol{s}$ be a relation on set B. Define relations $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ on $A \times B$ as follows:

$$(a_1, b_1)\boldsymbol{t}_1(a_2, b_2) \;\leftrightarrow\; [a_1\boldsymbol{r}a_2 \;\wedge\; b_1\boldsymbol{s}b_2],$$

$$(a_1, b_1)\boldsymbol{t}_2(a_2, b_2) \;\leftrightarrow\; [(a_1 \neq a_2 \;\wedge\; a_1\boldsymbol{r}a_2) \;\vee\; (a_1 = a_2 \;\wedge\; b_1\boldsymbol{s}b_2)].$$

46. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are reflexive, then $\boldsymbol{t}_1$ is reflexive.

47. If $\boldsymbol{s}$ is reflexive, then $\boldsymbol{t}_2$ is reflexive.

48. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are irreflexive, then $\boldsymbol{t}_1$ is irreflexive.

49. If $\boldsymbol{s}$ is irreflexive, then $\boldsymbol{t}_2$ is irreflexive.

50. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric, then $\boldsymbol{t}_1$ is symmetric.

51. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are symmetric, then $\boldsymbol{t}_2$ is symmetric.

52. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are antisymmetric, then $\boldsymbol{t}_1$ is antisymmetric.

53. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are antisymmetric, then $\boldsymbol{t}_2$ is antisymmetric.

54. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are transitive, then $\boldsymbol{t}_1$ is transitive.

55. If both $\boldsymbol{r}$ and $\boldsymbol{s}$ are transitive, then $\boldsymbol{t}_2$ is transitive.