Linear Programming

It's never been programming, and now it's not even linear!

William J. Martin

Department of Mathematical Sciences
and
Department of Computer Science
Worcester Polytechnic Institute

WPI Math Meet, October 19, 2011

Appreciative Graduates

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Math Diversions

Q: How much time does it save to cut diagonally across a square parking lot, park, or lawn?

Q: For beginning protractor users: Zoom in on Google maps and figure out the system for numbering runways at airports.

Q: What's wrong with this McGraw-Hill pre-calculus text?

> *To find horizontal asymptotes of a rational function $y = f(x)$, first solve for x and locate those values of y where this function is undefined.*

Linear Programming in the Sci-Fi Literature

'I don't want to bore you', Harvey said, 'but you should understand that these heaps of wire can practically think — linear programming — which means that instead of going through all the alternatives they have a hunch which is the right one.'

– Billion Dollar Brain, by Len Deighton

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- judge sent him to jail (for one day) until he proposed a new food plan
- Bartlett had pocketed \$212,000 in surplus meal monies in just 3 years
- $1.75 \times 300 \times 365 = \$191,625$ per year allowance

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Photo Credit: trusty guides

Photo Credit: stockfood.com

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Isn't math wonderful!

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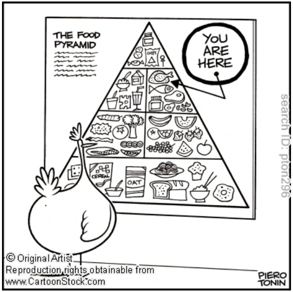

Photo Credit: cartoon stock.com

Here's a web interface to linear programming software that allows you to solve your own diet problems:

`http://www.neos-guide.org/NEOS/index.php/Diet_Problem_Demo`

A Few Simple Messages

- math majors get jobs (Jobs Rated 2011: top four professions are . . .)
- math is still alive, new theorems every year, many unsolved problems
- for example, linear programming is cool, powerful, versatile, changing
- In 1970, a study estimated that 25% of all computing was devoted to LP

Solving Systems of Linear Equations

A system of linear equations (3 equations, 3 unknowns):

$$\begin{array}{rcrcrcr} 3x & + & 5y & & & = & 18 \\ 2x & & & - & 5z & = & 2 \\ & & 2y & + & 3z & = & 6 \end{array}$$

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Any more solutions? Yes!

All solutions with $1 \leq t \leq 2$ are non-negative vectors. (Convexity!)

The Language of Linear Algebra

Let's revisit our system of 3 equations in 3 unknowns:

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & & & = & 18 \\ 2x_1 & & & - & 5x_3 & = & 2 \\ & & 2x_2 & + & 3x_3 & = & 6 \end{array}$$

This is written $\mathbf{Ax} = \mathbf{b}$ for

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 0 \\ 2 & 0 & -5 \\ 0 & 2 & 3 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 18 \\ 2 \\ 6 \end{bmatrix}.$$

The Basic Problem

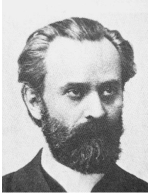
Given a matrix $\mathbf{A}$ with m rows and n columns and a vector $\mathbf{b}$ with m rows, we have the solution set of all vectors $\mathbf{x}$ of length n satisfying

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

Linear Programming: Find a solution with no negative entries.

This is equivalent to the more familiar formulation of maximizing some linear profit function subject to some collection of constraints which are linear equations or inequalities.

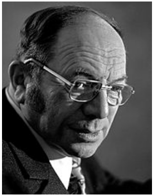
Farkas' Lemma



Gyula Farkas (1847-1930)

Theorem (1902): Given matrix $\mathbf{A}$ and vector $\mathbf{b}$
EITHER
there is a non-negative vector $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$
OR
there is a vector $\mathbf{y}$ such that $\mathbf{y}^\top \mathbf{A} \geq \mathbf{0}$ and yet $\mathbf{y}^\top \mathbf{b} < 0$
NOT BOTH.

History of Linear Programming



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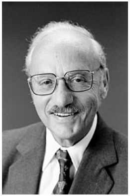

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- Operations Research
- Linear Programming (1939-1947)
- Simplex Algorithm (1947)
- Duality Theorem (1950)

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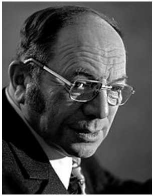

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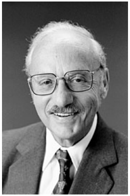

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- (Military) **Operations** Research
- Linear **Program**ming (a **program** is a plan of action)
- Simplex Algorithm
- Duality Theorem (rediscovery of Farkas' Lemma)

Polynomial Time Algorithms are "Fast"

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- E.g., sorting, shortest path are polynomial time; graph coloring, traveling salesman are probably exponential
- **Q:** If you can verify a YES answer in polynomial time and you can verify a NO answer in polynomial time, can't you decide whether the answer is YES or NO in polynomial time? (I.e., is $P = NP \cap coNP$?)
- Big question in the 1970s: Can linear programming problems be solved in polynomial time?

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- **Worse News:** numerically unstable in higher dimensions

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- ideas from non-linear optimization applied to linear problem
- log-barrier method, Newton-type method with projective (non-linear) scaling
- running time (according to wikipedia) is $O(n^{7/2}\ L^2\ \log L\ \log\log L)$ for a problem with n variables and L bits of input
- Very complicated, but similar to Affine Scaling Method, which is easy to teach

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- Anecdote: AT&T applied this algorithm to optimize their Pacific basin network
- I was told that this saved the company $20 million per year in operating costs

A Flurry of Activity to this Day

- Interior point methods have been a very active area of research since Karmarkar's result
- We now have polynomial time algorithms for many types of "conic programming" problems
- Most importantly, we have efficient algorithms for **semidefinite programming**
- SDP encompasses LP, but also includes special types of quadratic constraints
- many applications in finance, graph theory, control theory, ...

Game Theory

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Our optimal strategy is to play **R**, **S**, **P** at random, each with 1/3 probability.

But what if the payoff for winning with Rock or Scissors is \$1 and the payoff for winning with Paper is \$2?

Q: What is the optimal strategy now?

Asymmetric Rock/Paper/Scissors

Setting: We play Rock/Paper/Scissors many times. The payoff for winning with Rock or Scissors is \$1;
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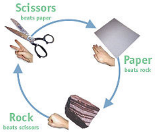

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Optimal Solution:

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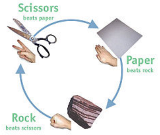

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Optimal Solution: $x_R = x_P = 1/4, \quad x_S = 1/2$

LP really is Artificial Intelligence

Hanshin Expressway, Osaka Japan

- very expensive toll road through a congested area
- Light sensors every 500m measure traffic volume
- A computer solves a linear programming problem every 15 minutes
- Decides which on-ramps to open and close
- Objective: to maximize number of vehicles subject to no traffic jams

The End

Good luck to all your Math Meet competitors!