Cometric Association Schemes

a survey

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Outline

Outline

Motivation, Definition, Examples

- Definitions and Parameters
- The Bannai-Ito Conjectures
- Extremal designs
- The known examples
- Imprimitivity

Imprimitive Cometric Schemes

- The Q-bipartite Case
- The Q-antipodal Case
- Dismantlability

Outline

The Leech lattice

- even unimodular lattice in $\mathbb{R}^{24}$
- kising number 196,560 (optimal)
- automorphism group is Co_1 (order 8,315,553,613,086,720,000)

We will focus on the spherical code consisting of the 196,560 (scaled) shortest vectors.

Shortest vectors

The 196,560 norm two vectors:

- $\frac{1}{\sqrt{8}}(\pm 2^8, 0^{16})$ — support is also support of a Golay codeword, even num of $-$ signs $(2^7 \cdot 759)$
- $\frac{1}{\sqrt{8}}(\mp 3, \pm 1^{23})$ — upper signs taken on support of a Golay codeword $(2^{12} \cdot 24)$
- $\frac{1}{\sqrt{8}}(\pm 4^2, 0^{22})$ — any two positions, any signs $(\binom{24}{2} \cdot 2^2)$

Scale these to unit vectors to get $X \subset S^{23}$.
Among these vectors, there are only 6 non-zero angles.

A curious property

There exists a function (a bilinear map)

$$\star : \mathbb{R}[t] \times \mathbb{R}[t] \longrightarrow \mathbb{R}[t]$$

for which

$$\sum z \in X f\left(\langle x, z\rangle\right) g\left(\langle z, y\rangle\right) = (f \star g)\left(\langle x, y\rangle\right)$$

for all polynomials f and g and for all $x, y \in X$.

Multiplication table

$\star$	1	t	t^2	t^3	t^4
1	196560				
t		$8190t$			
t^2	8190		$630t^2+315$		
t^3		$945t$		$\frac{135}{2}t^3+\frac{405}{4}t$	
t^4	945		$135t^2+\frac{135}{4}$		$9t^4+27t^2+\frac{27}{8}$
t^5		$\frac{675}{4}t$		$\frac{45}{2}t^3+\frac{135}{8}t$	
t^6	$\frac{675}{4}$		$\frac{135}{4}t^2+\frac{45}{8}$		$\frac{135}{32}t^4+\frac{405}{64}t^2+\frac{135}{256}$

Multiplication table, cont'd

$$t^5 \star t^5 = \frac{45}{32} t^5 + \frac{225}{32} t^3 + \frac{675}{256} t$$

$$t^6 \star t^6 = \frac{45}{32} t^6 + \frac{675}{512} t^4 + \frac{1485}{1024} t^2 + \frac{315}{4096}$$

Non-standard definition

A finite set X of points on the unit sphere in $\mathbb{R}^m$ is an *association scheme* if there exists a function

$$\star : \mathbb{R}[t] \times \mathbb{R}[t] \longrightarrow \mathbb{R}[t]$$

for which

$$\sum z \in X f\left(\langle x, z\rangle\right) g\left(\langle z, y\rangle\right) = \left(f \star g\right)\left(\langle x, y\rangle\right)$$

for all polynomials f and g and for all $x, y \in X$.

Non-standard definition

Supppose the finite set X of points on the unit sphere in $\mathbb{R}^m$ is an association scheme according to the above definition. The scheme is said to be *cometric* (or "Q-polynomial") if, for all polynomials f and g,

$$\deg f \star g \leq \; \deg f, \; \deg g$$

Example

The m-cube is a cometric association scheme in $\mathbb{R}^m$.
For $m = 3$, we have

$$1 \star 1 = 8$$

$$t \star t = \frac{8}{3} t$$

$$t^2 \star 1 = \frac{8}{3} \qquad t^2 \star t^2 = \frac{16}{9} t^2 + \frac{8}{27}$$

$$t^3 \star t = \frac{56}{27} t \qquad t^3 \star t^3 = \frac{16}{9} t^3 + \frac{56}{243} t$$

Outline

Association Schemes

Formally, a (symmetric) *association scheme* consists of a set X and a set $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ of relations on X with

- $\mathcal{R}$ is a partition of $X \times X$
- $R_0 = \mathrm{id}_X$ is the identity relation
- $R_i^\top = R_i$ (each relation is symmetric)
- for each i, j, k there exists an integer p_{ij}^k such that

$$|\{c \in X \mid (a,c) \in R_i,\ (c,b) \in R_j\}| = p_{ij}^k$$

whenever $(a,b) \in R_k$.

Bose-Mesner algebra

Let A_i be the zero-one adjacency matrix of R_i. Then

$$\mathcal{A} = \operatorname{span}\{A_0, \ldots, A_d\}$$

is a commutative semisimple matrix algebra containing I.
It is also closed entrywise multiplication $\circ$ (also called "Schur mult." or "Hadamard mult.") and contains the identity J for this multiplication.

Orthogonality relations

$$A_i = \sum_{j=0}^{d} P_{ji} E_j \qquad E_j = \frac{1}{v} \sum_{i=0}^{d} Q_{ij} A_i$$

The change-of-basis matrices P and Q are called the "first and second eigenmatrices" of the scheme. A scaled version of P is called the "character table":

$$PQ = vI$$

$$MP = Q^\top K$$

where M is a diagonal matrix of multiplicities $m_j = \mathrm{rank}\, E_j$ and K is a diagonal matrix of valencies $v_i = \text{rowsum} A_i$.

A taste of duality

$$A_iA_j = \sum_{k=0}^{d} p_{ij}^k A_k \qquad E_i \circ E_j = \frac{1}{v}\sum_{k=0}^{d} q_{ij}^k E_k$$

$$A_i \circ A_j = \delta_{ij} A_i \qquad E_iE_j = \delta_{ij} E_i$$

$$A_iE_j = P_{ji} E_j \qquad A_i \circ E_j = \frac{1}{v} Q_{ij} A_i$$

$$\sum_{i=0}^{d} A_i = J \qquad \sum_{j=0}^{d} E_j = I$$

$$A_0 = I \qquad E_0 = \frac{1}{v} J$$

Metric and Cometric Schemes

The scheme is *metric* (or *P-polynomial*) if there is an ordering of the A_i for which

- $p_{ij}^{k} = 0$ whenever $k > i + j$
- $p_{ij}^{i+j} > 0$ whenever $i + j \leq d$

The scheme is *cometric* (or *Q-polynomial*) if there is an ordering of the E_j for which

- $q_{ij}^{k} = 0$ whenever $k > i + j$
- $q_{ij}^{i+j} > 0$ whenever $i + j \leq d$

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The Conjectures of Bannai and Ito

Conjecture (Bannai & Ito)

Every primitive cometric scheme of sufficiently large diameter d is metric as well.

The Conjectures of Bannai and Ito

Let $V_j = \mathrm{colsp} E_j$ denote the j^{th} eigenspace of the cometric scheme.

Conjecture (Bannai & Ito)

The multiplicities $m_0, m_1, \ldots, m_d$ *of a cometric association scheme, given by* $m_j = \dim V_j$ *form a unimodal sequence:*

$$m_0 < m_1 \leq m_2 \leq \cdots \leq m_r \geq m_{r+1} \geq \cdots \geq m_d.$$

The Conjectures of Bannai and Ito

Conjecture (D. Stanton)

For $j < d/2$,

$$m_j \leq m_{j+1}, \qquad m_j \leq m_{d-j}.$$

Theorem (Caughman & Sagan, 2001)

If $(X, \mathcal{R})$ *is also* dual thin, *then Stanton's conjecture holds.*

Dual Thin Cometric Schemes

Call a subspace $W \subseteq \mathbb{R}^X$ $\mathcal{A}$-*invariant* if

$$\dim E_j W = \dim(V_j \cap W)$$

for all j (I.e., W admits a basis of eigenvectors).
A $\mathcal{T}$-*module* (with respect to basepoint 1) is an $\mathcal{A}$-invariant subspace which is closed under multiplication by Z_1 (i.e., Schur multiplication by first column of E_1).
A $\mathcal{T}$-module W is *dual thin* if $\dim(V_j \cap W) \leq 1$ for all j.
The cometric scheme $(X, \mathcal{R})$ is *dual thin* if every irreducible $\mathcal{T}$-module for that scheme is dual thin.

Outline

Spherical Designs

Spherical t-Design: Finite subset $X \subset S^{m-1}$ for which

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{\| S^{m-1} \|} \int f(x) dx$$

for all polynomials f in m variables of total degree at most t.

Example: The 196,560 shortest vectors of the Leech lattice form a spherical 11-design in $\mathbb{R}^{24}$.

Seymour and Zaslavsky (1984): Such finite sets X exist for all t in each dimension m.

Cometric schemes from spherical designs

Theorem (Delsarte,Goethals,Seidel (1977))

The number s of non-zero angles in a spherical t-design is at least $t/2$. If $t \geq 2s-2$, then X carries a cometric association scheme.

Examples: 24-cell ($t=5$, $s=4$); E_6 ($t=5$, $s=4$); E_8 ($t=7$, $s=4$); Leech ($t=11$, $s=6$).

Cometric schemes from combinatorial designs

Definition: A *Delsarte t-design* in a cometric scheme (X, A) is any non-trivial subset C of X whose characteristic vector x_C is orthogonal to $V_1, \ldots, V_t$.

Examples: orthogonal arrays ("dual codes"), block designs.

Theorem (Delsarte (1973))

If s non-zero relations occur among pairs of elements of C, then $t \leq 2s$. If $t \geq 2s - 2$, then C carries a cometric association scheme.

Cometric schemes from semilattices

Definition: The *dual width* w^* of $C \subseteq X$ is the maximum j for which $E_j x_C \neq 0$.

Theorem (Brouwer, Godsil, Koolen, WJM (2003))

For any C *in a* d*-class cometric scheme,* $w^* \geq d - s$*. If equality holds, then* C *carries a cometric association scheme.*

Group schemes

Every finite group G yields an association scheme via the center of the group algebra of its right regular representation $g \mapsto R_g$.
Conjugacy classes: $\mathcal{C}_0 = \{e\}, \mathcal{C}_1, \ldots, \mathcal{C}_n$

$$A_i = \sum_{g \in \mathcal{C}_i} R_g$$

Extended conjugacy classes: $\mathcal{C}'_0 = \{e\}, \mathcal{C}'_i = \mathcal{C}_i \cup (\mathcal{C}_i)^{-1}$
Symmetrized scheme:

$$A_i = \sum_{g \in \mathcal{C}'_i} R_g$$

Cometric group schemes

Theorem (Kiyota and Suzuki (2000))

The symmetrized group scheme is cometric if and only if G is one of the following groups:

- $\mathbb{Z}_n$
- S_3
- A_4
- $SL(2,3)$
- F_{21}

Outline

A Census

The following cometric association schemes are known:

- Q-polynomial distance-regular graphs (i.e., metric and cometric)
- duals of metric translation schemes
- bipartite doubles of Hermitian forms dual polar spaces $[{}^2A_{2d-1}(r)]$ (Bannai & Ito)
- schemes arising from linked systems of symmetric designs (3-class, Q-antipodal) [Cameron & Seidel]
- extended Q-bipartite doubles of linked systems (4-class, Q-bipartite and Q-antipodal) [Muzychuk, Williford, WJM (2007)]

Census,cont'd

- the block schemes of the Witt designs 4-(11,5,1), 5-(24,8,1) and a 4-(47,11,8) design (Delsarte) (primitive 3-class schemes on 66, 759 and 4324 vertices resp.)
- the block schemes of the 5-(12,6,1) design and the 5-(24,12,48) design (Q-bipartite 4-class schemes on 132 and 2576 vertices, resp.)
- shortest vectors in lattices E_6, E_7, E_8 (4-class, Q-bipartite)
- the scheme on the vertices of the 24-cell (4-class, Q-bipartite, Q-antipodal, 24 vertices)

Census, cont'd

- the scheme on the shortest vectors in the Leech lattice (6-class, Q-bipartite, 196560 vertices)
- 5 schemes arising from derived designs of this:

3-class	2025 vertices	primitive
4-class	2816	Q-bipartite
4-class	4600	Q-bipartite
4-class	7128	primitive
5-class	47104	primitive

- antipodal (or Q-bipartite) quotient of Leech lattice example (3-class, primitive)
- three more schemes arising from lattices (4-, 5-, 11-class, Q-bipartite)
- three schemes from dismantling dual schemes of metric translation schemes (4-, 5-, and 6-class, all Q-antipodal)

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- Imprimitive Cometric Schemes
  - The Q-bipartite Case
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  - Dismantlability
- The Q-Ideal
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  - Small degree
  - Lewis's Homotopy
  - Conjectures

Imprimitivity

An association scheme is *imprimitive* if there is a subset $A_{i_0}, A_{i_1}, \ldots, A_{i_e}$ of the associate matrices A_i satisfying $\sum_h A_{i_h} = I_w \otimes J_r$ for some $1 < w, r < v$.

Any imprimitive distance-regular graph is either bipartite or antipodal or both.

Imprimitivity

Theorem (Suzuki, 1998)

Any imprimitive cometric association scheme is either Q-bipartite or Q-antipodal or both, with possible exceptions if the number of classes is four or six.

Theorem (Cerzo and Suzuki, 2006)

The exception with $d = 4$ *does not occur.*

Outline

Q-bipartite Schemes

The parameters of a cometric scheme $(X, \mathbf{A})$ are entirely determined by its *Krein array*

$$\iota^*(X, \mathbf{A}) = \{b_0^*, b_1^*, \ldots, b_{d-1}^*; c_1^*, c_2^*, \ldots, c_d^*\}$$

where $b_j^* = q_{1,j+1}^j$, $c_j^* = q_{1,j-1}^j$ and we also define $a_j^* = q_{1j}^j = m_1 - b_j^* - c_j^*$.
A cometric scheme is *Q-bipartite* if all $a_j^* = 0$. This is equivalent to the condition that $q_{ij}^k = 0$ whenever $i + j + k$ is odd.

Q-bipartite Structure

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- **Theorem:** A Q-bipartite scheme has $r = 2$ (Brouwer,Godsil, Koolen,WJM)
- With natural ordering, sequence of cosines Q_{i1} is symmetric about the origin
- $p_{ij}^{k} = p_{i,d-j}^{d-k}$ for all $0 \leq i, j, k \leq d$.

Outline

Q-antipodal Structure

- **Gardiner, 1970s:** a P-antipodal scheme has $r \leq k$

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Q-antipodal Structure

- **Gardiner, 1970s:** a P-antipodal scheme has $r \leq k$
- **Theorem:** A Q-antipodal scheme with d odd has $w \leq m_1$ (d even case not complete)
- With natural ordering, $Q_{0d} = Q_{2d} = \cdots = m_d$ and $Q_{1d} = Q_{3d} = \cdots = -1$
- $p_{ij}^k = 0$ unless $i + j + k$ is even **or** ijk odd.

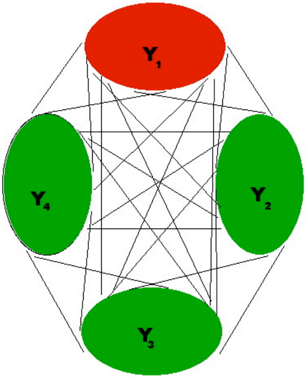
Outline

Dismantlability

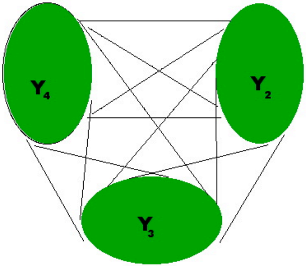
Theorem (Muzychuk, Williford, WJM (2007))

Every Q-antipodal scheme is dismantlable:
the subscheme induced on any non-trivial collection of w' *Q-antipodal classes is cometric for* $w' \geq 1$ *and Q-antipodal with* d *classes for* $w' > 1$.

Dismantlability



Dismantlability



Trivial cases

- halved graph of a bipartite Q-polynomial distance-regular graph
- linked systems of symmetric designs (by defn.)

A new example via dismantling

Coset graph of the shortened ternary Golay code:

- intersection array $\{20, 18, 4, 1; 1, 2, 18, 20\}$

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- Remove one of these to obtain a Q-antipodal scheme on 162 vertices having $w = 2$ Q-antipodal classes which is not metric

A new example via dismantling

Coset graph of the shortened ternary Golay code:

- intersection array $\{20, 18, 4, 1; 1, 2, 18, 20\}$
- antipodal distance-regular graph belonging to a translation scheme
- dual association scheme is Q-antipodal on $v = 243$ vertices with $w = 3$ Q-antipodal classes
- Remove one of these to obtain a Q-antipodal scheme on 162 vertices having $w = 2$ Q-antipodal classes which is not metric
- parameters

$$d = 4, \;\; v = 162, \;\; \iota^*(X, \mathbf{A}) = \{20, 18, 3, 1; 1, 3, 18, 20\}$$

formally dual to those of an unknown diameter four bipartite distance-regular graph.

Dismantling the dual of a coset graph

- Two more distance-regular coset graphs yield Q-antipodal schemes with five and six classes.

Dismantling the dual of a coset graph

- Two more distance-regular coset graphs yield Q-antipodal schemes with five and six classes.
- Parameters

  $d = 5,\ v = 486,$

  $$\iota^*(X, \mathbf{A}) = \{22, 20, \frac{27}{2}, 2, 1;\ 1, 2, \frac{27}{2}, 20, 22\},\ \ w = 2$$

  $d = 6,\ v = 1536,$

  $$\iota^*(X, \mathbf{A}) = \{21, 20, 16, 8, 2, 1;\ 1, 2, 4, 16, 20, 21\},\ \ w = 3.$$

Dismantling the dual of a coset graph

- Two more distance-regular coset graphs yield Q-antipodal schemes with five and six classes.
- Parameters
  $d = 5,\ v = 486,$
  $$\iota^*(X, \mathbf{A}) = \{22, 20, \frac{27}{2}, 2, 1;\ 1, 2, \frac{27}{2}, 20, 22\},\ \ w = 2$$
  $d = 6,\ v = 1536,$
  $$\iota^*(X, \mathbf{A}) = \{21, 20, 16, 8, 2, 1;\ 1, 2, 4, 16, 20, 21\},\ \ w = 3.$$
- This last scheme is formally dual to a distance-regular graph which was proven not to exist by Brouwer, Cohen and Neumaier.

Building Up?

- **Theorem (Muzychuk,Williford,WJM):** The first multiplicity does not depend on w
- **Question:** Which Q-polynomial bipartite drg's can be extended to $w > 2$?
- Works for 2-cube, 3-cube, 4-cube

A Characterization of Linked Systems

Corollary (van Dam)

Every Q-antipodal 3*-class cometric association scheme arises from a linked system of symmetric designs.*

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The 4-cycle

$$E_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Ring homomorphism $\gamma : \mathbb{R}[Z_1, Z_2, Z_3, Z_4] \to \mathbb{R}^4$ takes

$$Z_1 \mapsto \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \qquad Z_2 \mapsto \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{etc.}$$

The 4-cycle

$$E_1 = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Ring homomorphism $\gamma : \mathbb{R}[Z_1, Z_2, Z_3, Z_4] \to \mathbb{R}^4$ takes

$$4Z_1 + 2Z_2 \mapsto \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \qquad Z_1 Z_2 \mapsto 0, \quad Z_1 Z_4 \mapsto \frac{1}{4} \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \text{etc.}$$

An elementary ring homomorphism

In general, let $(X, \mathcal{R})$ be a cometric association scheme on v vertices with first primitive idempotent E_1.
Let $\gamma : \mathbb{R}[Z_1, \ldots, Z_v] \to \mathbb{R}^X$ via

$$Z_a \mapsto \bar{a}$$

(the a-column of E_1) and extending linearly and via the Schur product $\circ$.
E.g., $Z_a Z_b^2 - 3Z_a \mapsto (\bar{a} \circ \bar{b} \circ \bar{b}) - 3\bar{a}$
We are interested in $I = \ker \gamma$.

The Q-Ideal

Object of study: $I = \ker \gamma$

Theorem

I is the set of polynomials in $\mathbb{R}[Z_1, \ldots, Z_v]$ which vanish on each column of E_1

Here, $v = |X|$ is the number of vertices in the cometric scheme $(X, \mathcal{R})$. Equivalently, we can look at an ideal I_N in the ring $\mathbb{R}[Y_1, \ldots, Y_{m_1}]$.

Outline

Small Degree Generators

In the n-cube, the code $C = \{ a \mid a_1 = 0 \}$ has width $n-1$ and dual width $w^* = 1$.
This gives a quadratic polynomial in our ideal:

$$F = \left(\sum_{c \in C} Z_c - \frac{1}{2} \right) \left(\sum_{c \in C} Z_c + \frac{1}{2} \right)$$

As C ranges over the dim. $n-1$ subcubes, this gives a set of quadratic polynomials which generate I_N.

Small Degree Generators

Likewise, the ideal for any Hamming or Johnson scheme is generated by linear and quadratic polynomials.

More Small Degree Generators

- 24-cell: I generated by polys. of degree at most four
- E_6: " degree at most three
- E_8: " degree at most four
- E_7: " degree at most four (maybe three?)
- Leech lattice: computations not finished

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Homotopy

Let Γ be a distance-regular graph (metric association scheme) and let x be any vertex. Equivalence classes of closed walks in Γ beginning and ending at x form a group under concatenation and reversal.

This is the *fundamental group* $\pi(\Gamma, x)$ of Γ and essentially does not depend on x.

A Sequence of Homotopy Groups

H. Lewis (2000):
The *essential length* of a walk w of the form pqp^{-1} is at most the length of walk q.
Definition: Let $\pi(\Gamma, x, k)$ be the subgroup of $\pi(\Gamma, x)$ generated by equivalence classes of closed walks of essential length at most k.

Theorem (Lewis)

If Γ *is a distance-regular graph of diameter* d*, then*
$\{e\} = \pi(\Gamma, x, 0) = \pi(\Gamma, x, 1) = \pi(\Gamma, x, 2) \subseteq$

$$\cdots \subseteq \pi(\Gamma, x, 2d+1) = \pi(\Gamma, x).$$

Translation Schemes

A translation scheme is a scheme $(X, \mathcal{R})$ where X is a finite abelian group and $(a, b) \in R_i$ implies $(a + c, b + c) \in R_i$.

We assume $(X, \mathcal{R})$ is a cometric translation scheme and then there is a distance-regular graph Γ defined on the group $X^\dagger$ of characters of X.

Some set S_1 of characters forms a basis for the first eigenspace in the Q-polynomial ordering of $(X, \mathcal{R})$. The graph has edges $(\psi, \psi \circ \chi)$ for $\chi \in S_1$.

So if $S_1 = \{\chi_1, \ldots, \chi_m\}$, then each walk $w = \psi_0, \psi_1, \ldots$ in Γ can be described by giving its starting point ψ_0, together with a sequence $h_1, h_2, \ldots, h_s$ for which $\psi_j = \psi_{j-1} \circ \chi_{h_j}$.

Homotopy and Duality

In a cometric translation scheme, each closed walk in the dual distance-regular graph Γ yields a polynomial in I_N and these generate I_N:

$$F_w = Y_{h_1} Y_{h_2} \cdots Y_{h_s} - 1$$

So if Lewis's subgroup $\pi(\Gamma, x, k)$ is the entire fundamental group $\pi(\Gamma, x)$, then the ideal I_N is generated by polynomials of total degree at most $(k+1)/2$.

Outline

Cycles are special

Here is a Gröbner basis for the ideal I_N (dimension two) in the case of the n-cycle:

$$X^2 + Y^2 - 1, \qquad (X-1)(X-\zeta_1)\cdots(X-\zeta_{\lfloor n/2 \rfloor})$$

where (with $\alpha = \frac{2\pi}{n}$) we have $\zeta_k = \cos(k\alpha)$.

Perhaps too bold

Conjecture

There is a universal constant K *such that, for any cometric association scheme with* $m_1 > 2$*, the ideal* I *is generated by polynomials of total degree at most* K.

A more reasonable conjecture

Conjecture

For each integer $m > 2$, *there is an integer* $K(m)$ *such that, for any cometric association scheme with* $\mathrm{rank}\, E_1 = m$, *the ideal* I *is generated by polynomials of total degree at most* $K(m)$.

Good news or bad?

The second conjecture is equivalent to the following dual of the Bannai-Ito Conjecture:

Conjecture

For any $m > 2$*, there are only finitely many cometric association schemes with* $\operatorname{rank} E_1 = m$*.*

Outline

Today's Main Result

This last conjecture is true!

Theorem

For any $m > 2$*, there are only finitely many cometric association schemes with* $\operatorname{rank} E_1 = m$*.*

So we have

Theorem

For each integer $m > 2$ *There is an integer* $K(m)$ *such that, for any cometric association scheme with* $\operatorname{rank} E_1 = m$*, the ideal* I *is generated by polynomials of total degree at most* $K(m)$*.*

Main Result

Theorem

For any $m > 2$, there are only finitely many cometric association schemes with $\operatorname{rank} E_1 = m$.

This part is joint work with Jason Williford.

Bounding the first valency

Lemma

Given E_1 *of rank* m*, let* R_1 *be the relation with* Q_{11} *largest among* $\{Q_{i1}|i > 0\}$ *(nearest neighbor relation). Then the valency* v_1 *is bounded above by the kissing number in dimension* $m-1$*.*

(Observed independently by Bannai and Bannai, 2006.)

Notation: So $v_1 \leq L$ for any cometric scheme with $\mathrm{rank}\, E_1 = m$.

Bounding the number of algebraic integers

CLAIM: There are only finitely many algebraic integers in the interval $[-L, L]$.

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FALSE!

Bounding the number of algebraic integers

CLAIM: For fixed n, there are only finitely many algebraic integers in the interval $[-L, L]$ which have degree at most n over the rationals.

Bounding the number of algebraic integers

CLAIM: For fixed n, there are only finitely many algebraic integers in the interval $[-L, L]$ which have degree at most n over the rationals.

FALSE!

Bounding the number of algebraic integers

CLAIM: For fixed n, there are only finitely many algebraic integers which have degree at most n over the rationals and all of whose algebraic conjugates lie in the interval $[-L, L]$.

Bounding the number of algebraic integers

CLAIM: For fixed n, there are only finitely many algebraic integers which have degree at most n over the rationals and all of whose algebraic conjugates lie in the interval $[-L, L]$.

TRUE!

The splitting field

Theorem (H. Suzuki, 1998)

A cometric association scheme admits at most two Q-polynomial orderings.

It follows that

Theorem

The splitting field of any cometric association scheme is at most a degree two extension of the rationals.

(Independently proved by Hosoya and Suzuki.)

Infinitely many schemes?

If there are infinitely many cometric schemes with $\mathrm{rank}\, E_1 = m$, then by spherical code arguments, we can find a scheme with Q_{11} as close to m_1 as we like.

Getting Sufficiently Close

Let r be the maximum of $s/\lceil s \rceil$ over irrational algebraic integers in the range $[0, L]$ which are quadratic over $\mathbb{Q}$ and whose conjugates lie in $[-L, L]$. Ensure also that $r > (L-1)/L$.

Then find a scheme in this family with

$$m > Q_{11} > r\, m.$$

The contradiction

We chose $r > (L-1)/L$ to be at least the maximum of $s/\lceil s \rceil$ over irrational algebraic integers in the range $[0, L]$ which are quadratic over $\mathbb{Q}$ and whose conjugates lie in $[-L, L]$.

Then we found a scheme with $\operatorname{rank} E_1 = m$ and

$$m > Q_{11} > r\, m.$$

Orthogonality relation: $m_j P_{ji} = v_i Q_{ij}$
This gives

$$v_1 > P_{11} > r v_1$$

or $P_{11}/v_1 > r$ which contradicts our choice of r.

One more Example

For the E_8 association scheme (240 vertices in $\mathbb{R}^8$), our ideal is generated by polynomials of degree two and four, namely

$$Y_h Y_i (Y_j^2 - Y_k^2)$$

$(h, i, j, k$ distinct) and

$$Y_h Y_i Y_j Y_k - Y_{h'} Y_{i'} Y_{j'} Y_{k'}$$

for any bipartition of $\{1, \ldots, 8\}$ into two sets of size four, together with

$$Y_1^2 + \cdots + Y_8^2 - 1.$$