Hey! You Can't Do That With My Code!

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Outline

A New Era in Coding Theory

Is algebraic coding theory dead?

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New Efficient Error-Correcting Codes

- turbo codes
- belief propagation
- low-density parity check codes
- proof by simulation (now made rigorous)
- large minimum distance no longer needed

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- Coding Theory
- Design Theory

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A Hint of Delsarte Theory

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Theorem: The dual of any additive code with minimum distance d is an OA of strength $t = d - 1$.

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- This is **not** what I want to talk about today.

Part I: (T, M, S)-Nets

Using Codes to Estimate Integrals

If orthogonal arrays can be used to approximate Hamming space, can they also be used to approximate other spaces?

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- **2004+:** Improved bounds (Schmid/Schürer/Bierbrauer/Barg/Purkayastha/Trinker/Visentin)

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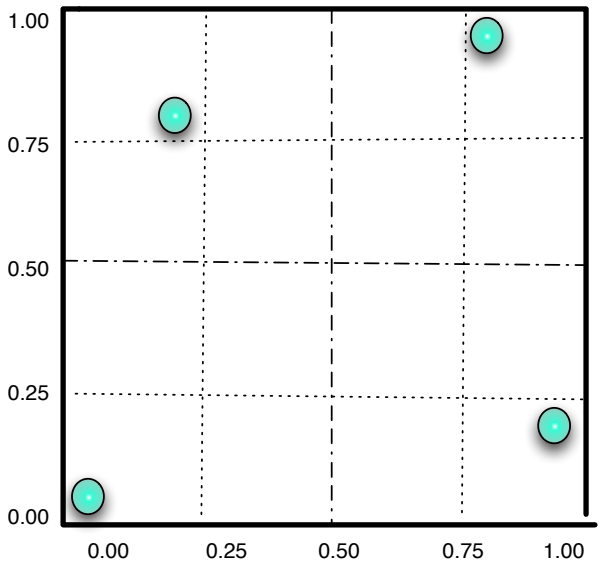
- 

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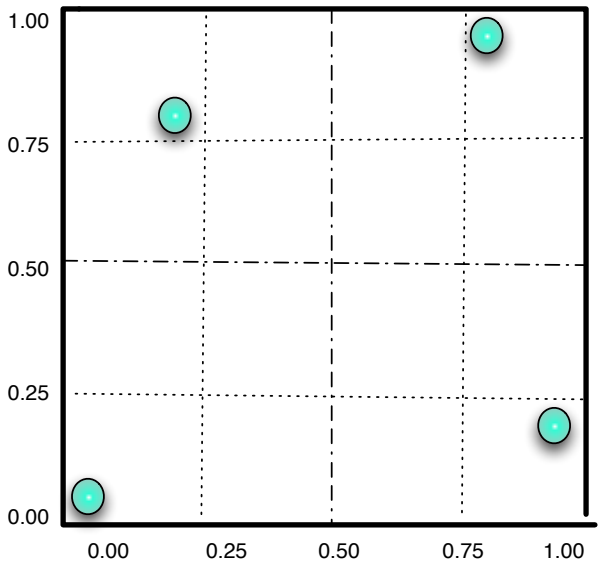

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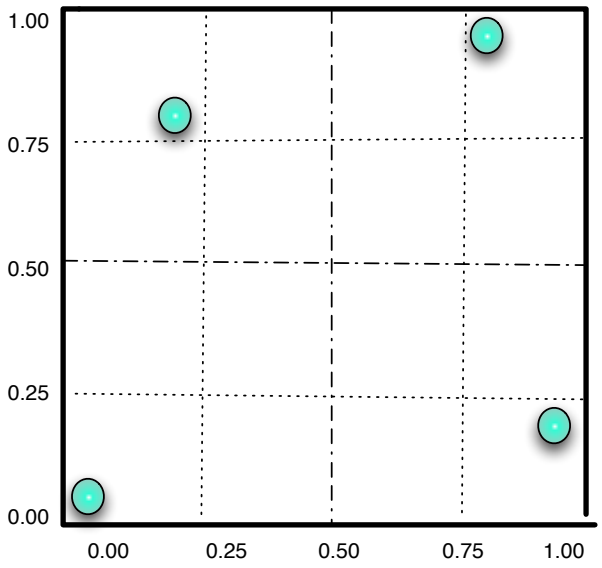

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- **Lawrence/Mullen/Schmid:** $\exists (T, M, S)$-net in base $q \Leftrightarrow \exists OOA$ over $\mathbb{F}_q$ with q^m rows, $s = S$, $\ell = t = M - T$.

The Theorem of Mullen & Schmid and (indep.) Lawrence

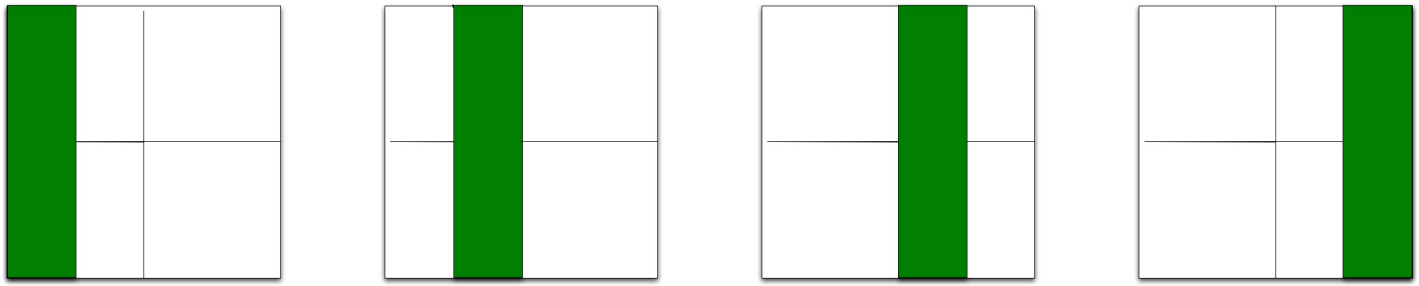


Theorem (1996): $\exists (T, M, S)$-net in base $q \Leftrightarrow \exists OOA$ over $\mathbb{F}_q$ with q^m rows, $s = S$, $\ell = t = M - T$

Idea of Proof

$$\mathcal{N} = \left\{\left(\frac{0}{4}, \frac{0}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{2}{4}, \frac{2}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right\}$$

$$T = \{(1,1), (1,2)\}$$



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$$\mathcal{N} = \{(.00, .00), (.01, .11), (.10, .10), (.11, .01)\}$$

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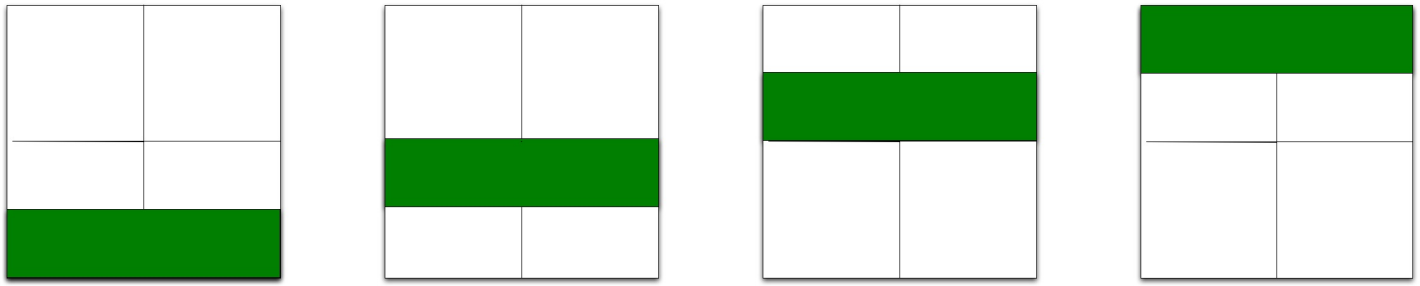

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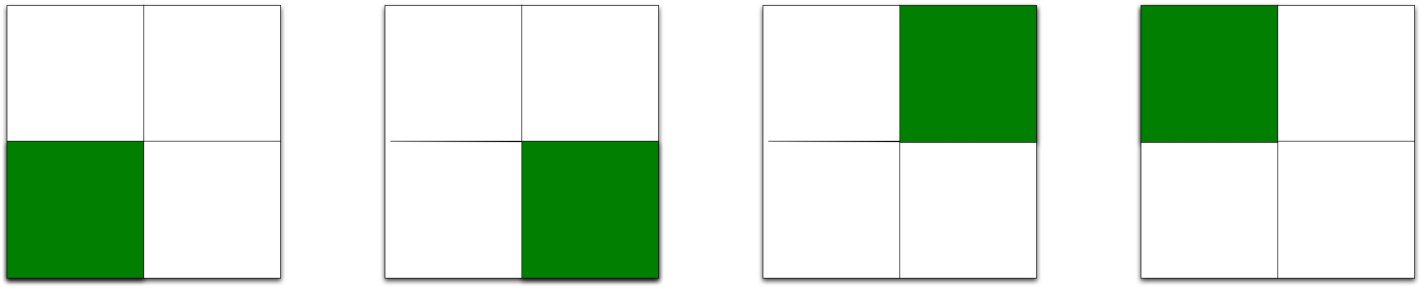
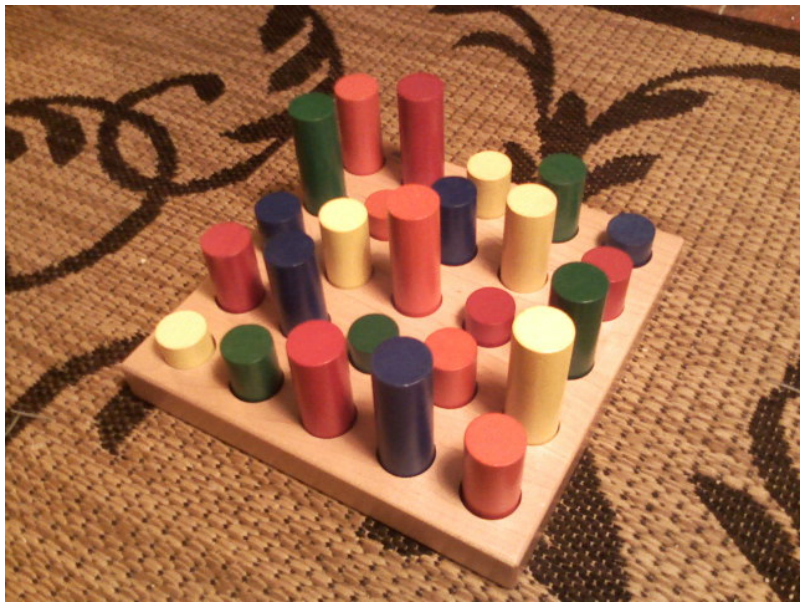
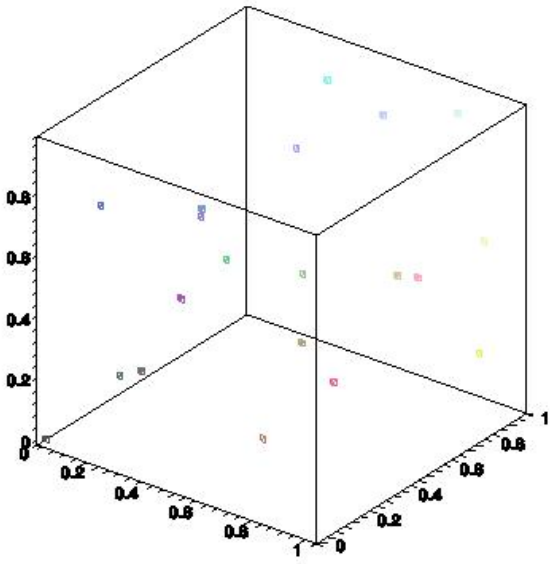


Nets from Many Sources





two mutually orthogonal latin squares of order five (color/height)

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- They show that the same works over algebraic curves (global function fields)

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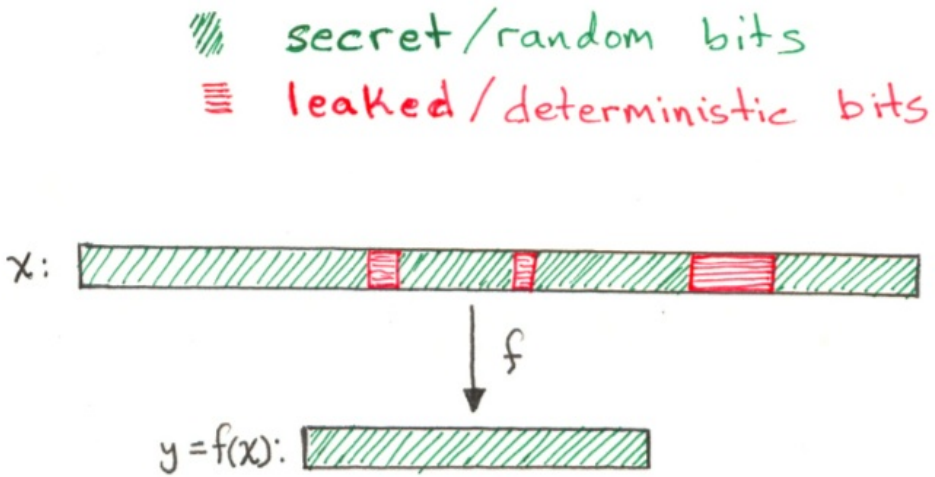
- **Research Problem:** Are there any non-trivial perfect codes in the Rosenbloom-Tsfasman metric?

Part II: Resilient Functions

Resilient Functions

How can a code be used to bolster randomness?

Resilient Functions



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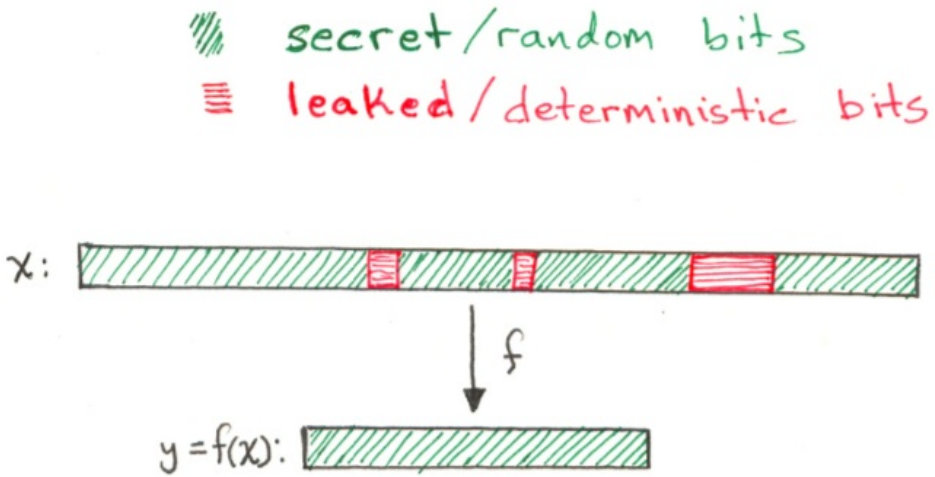

We have a secret string x. An opponent learns t bits of x, but we don't know which ones.
After applying function f, we guarantee that our opponents knows nothing.

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- **1999+:** Applications to fault-tolerant distributed computing, key distribution, quantum cryptography, etc.

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- **Why?** Any linear combination of entries of $f(x)$ is a dot product of x with some codeword
- So any non-trivial linear function of entries involves at least one random input position

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- What if quiet period is longer than expected?

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- **Lemma** (Sunar/WJM): Let X be a random variable taking values in $\{0,1\}^n$ according to a probability distribution $\mathcal{D}_{T,A}$. Then

$$\text{Prob}[H_{\text{out}} = k - r \mid |T| = i] \;=\; B_{i,r} \binom{n}{i}^{-1}.$$

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Can we work out these statistics for the other standard families of codes?

Part III: Fuzzy Extractors

Codes for Biometrics

How can we eliminate noise if we are not permitted to choose our codewords?

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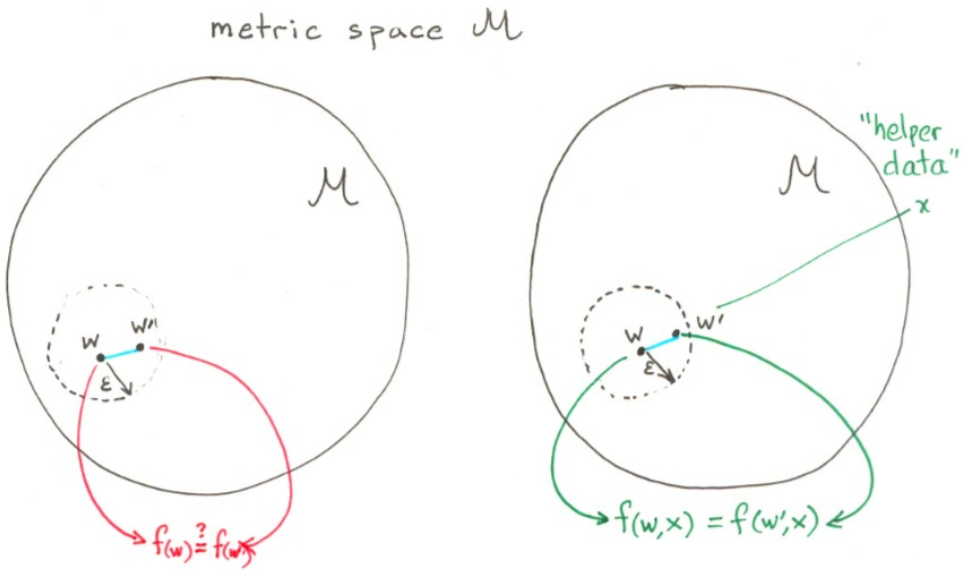
Fuzzy Extractors



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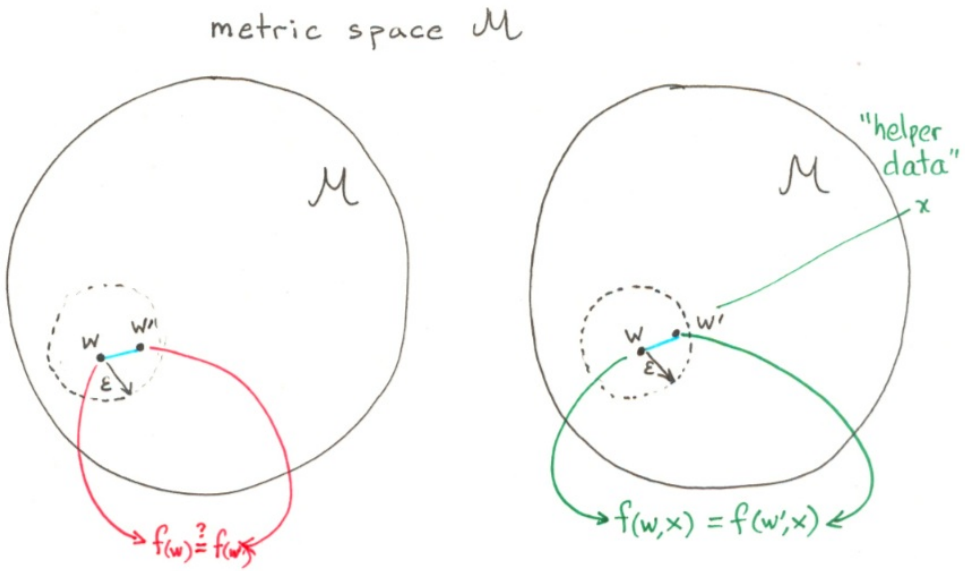
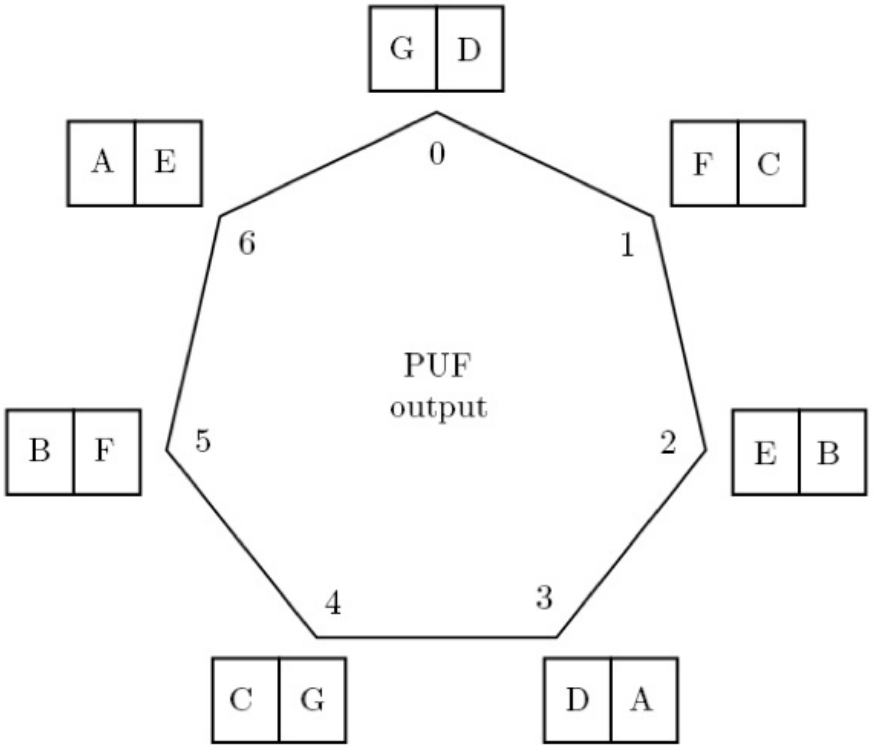


Metric space $\mathcal{M}$ and function $f : \mathcal{M} \times \{0,1\}^* \to \{0,1\}^*$ such that $f(w', x) = f(w, x)$ provided x valid for w and $d(w', w) < \epsilon$.

Fuzzy Extractor: Toy Example



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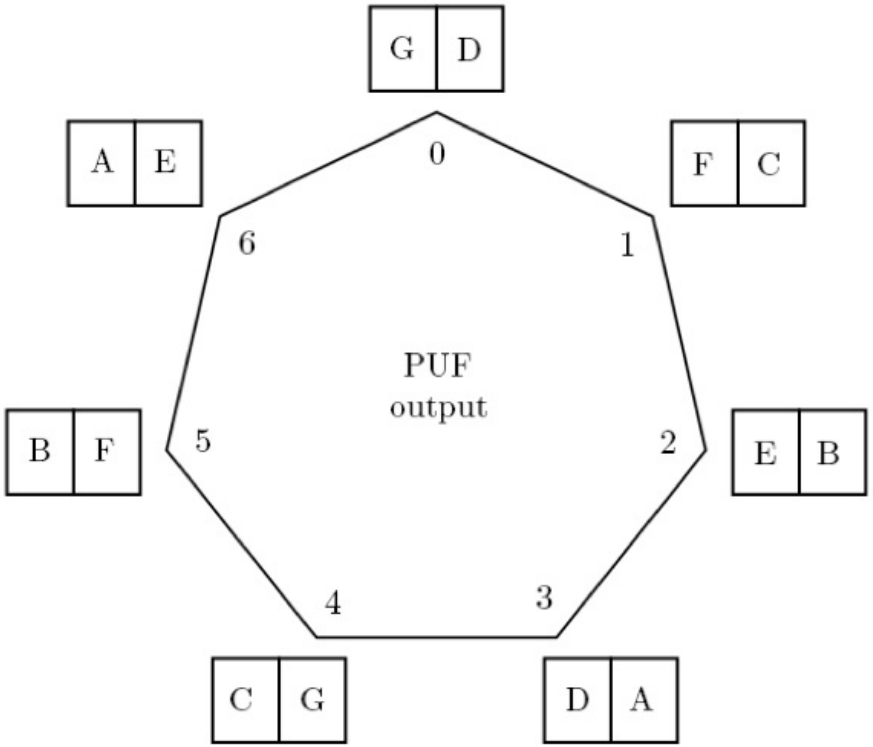

Baseline reading $w = 3$ is obtained from temporal reading $w' = 2$ and hint $x = D$.

But w is not recoverable from either w' or x alone.

Code-Offset Construction (Dodis, et al.)

Fuzzy extractor for Hamming metric:

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- To recover m from x and w', decode $w' + x = mG + (w - w')$
- Provided k and d are both linear in n, recovery of m from just x or w' is hard

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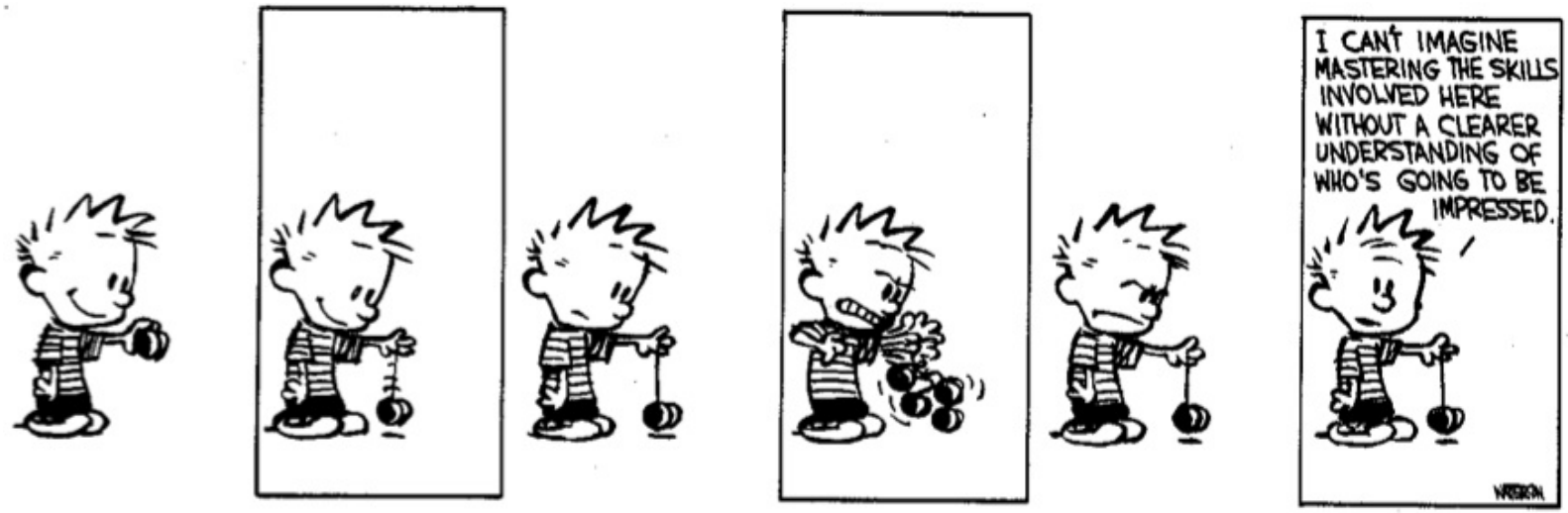
Can we build efficient fuzzy extractors for the Euclidean metric?

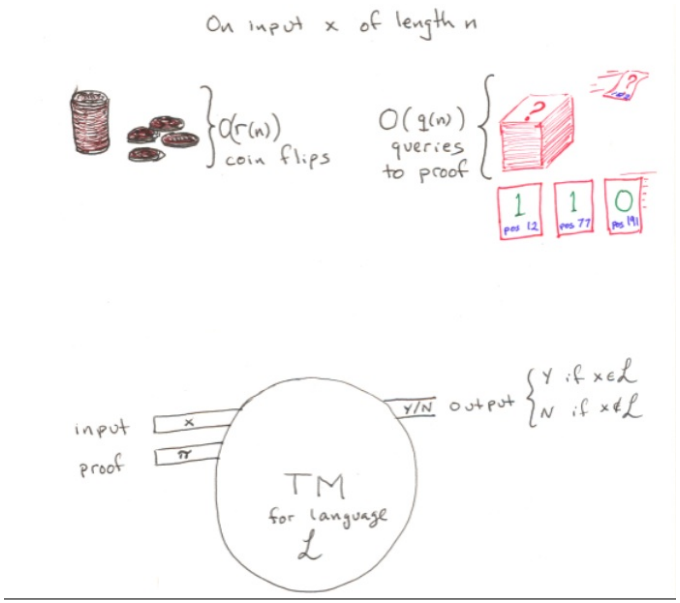
Part IV: The PCP Theorem



Codes and Computational Complexity

How does coding theory make computational problems more robust?

Probabilistically Checkable Proofs



Randomized Turing Machine uses $O(r(n))$ random bits and makes $O(q(n))$ queries to the proof π.

If π is a correct proof, x is accepted with probability one; if not, x is rejected with probability at least $\frac{1}{2}$.

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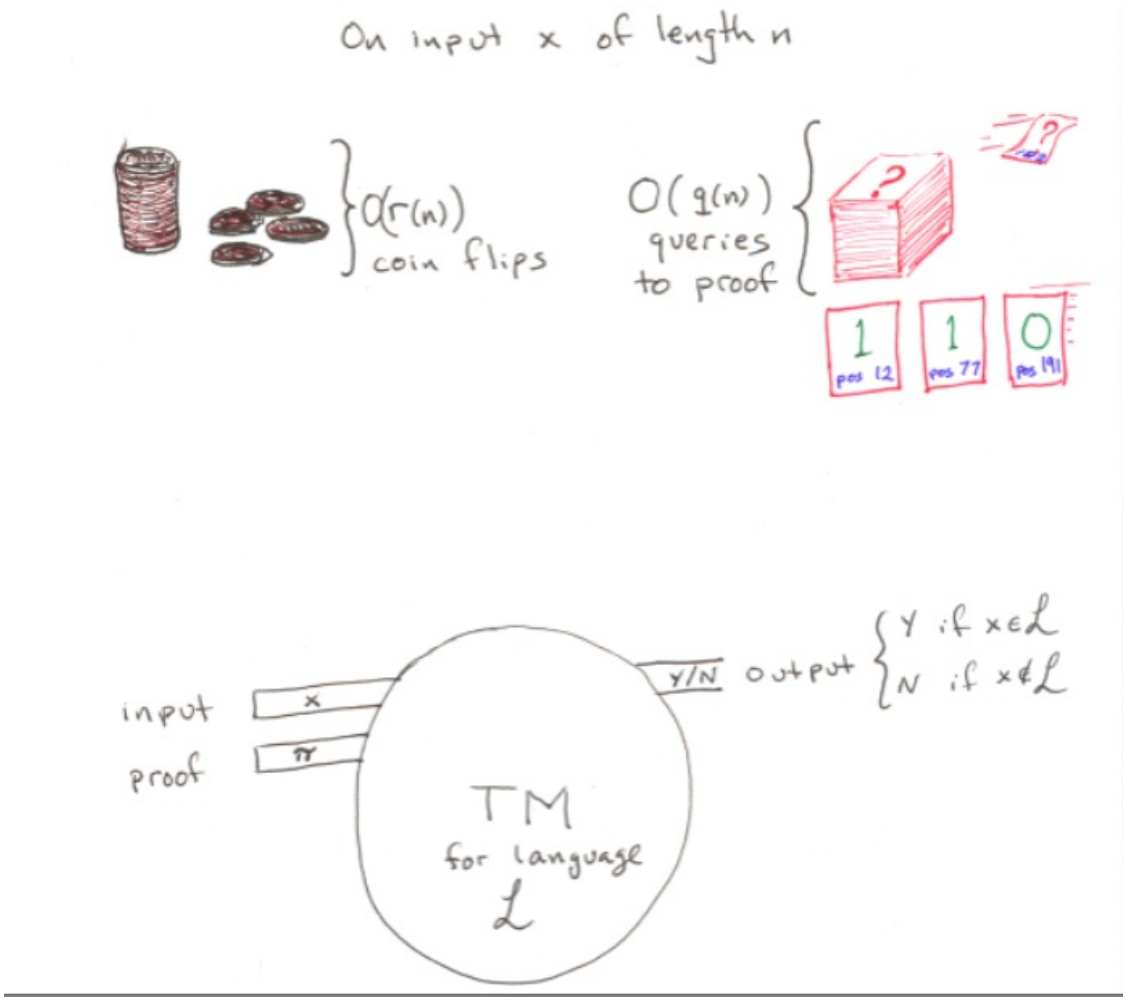

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- the set of languages for which such a polynomial time V exists is denoted $PCP[r(n), q(n)]$

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- $x : H^m \to \{0, 1\}$ where $H \subseteq \mathbb{F}_q$ has size $n^{1/m}$ has low-degree extension

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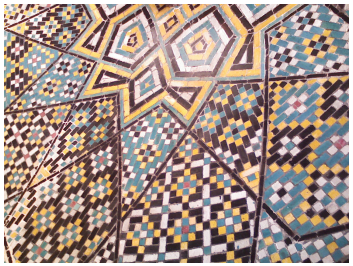
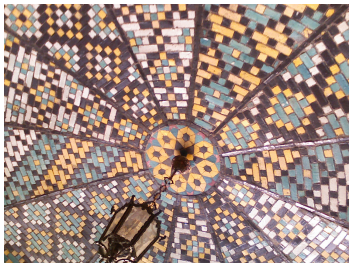
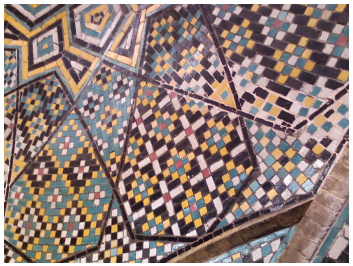
- For some $c > 1$, there is no n^c-approximate algorithm for Independent Set unless $\mathbf{P} = \mathbf{NP}$

The End


Thank you all!