An ideal associated to any cometric association scheme

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Outline

Why Association Schemes?

- CODING THEORY
- DESIGN THEORY

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- *"Distinguishability"*

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- *"Approximation"*

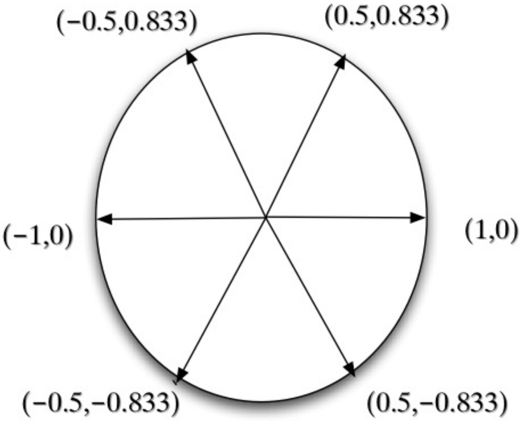
Why Association Schemes?

- CODING THEORY
- *"Distinguishability"*
- E.g., binary codes in Hamming scheme $H(n, q)$

- DESIGN THEORY
- *"Approximation"*
- E.g, t-(v, k, λ) designs

Six Vectors in $\mathbb{R}^2$

We will start by looking at a very simple example.



Spherical Code

A *spherical code* is simply a finite non-empty subset of the unit sphere.

$$X \subset S^{m-1}$$

(We'll set $v = |X|$ and assume $v > m$.)
Example: $m = 2, \ v = 6$

$$X = \left\{ (1,0), \ \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \ \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \ (-1,0), \ \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \ \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \right\}$$

Gram Matrix

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -1 & 0 \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & -1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & 2 & 1 & -1 & -2 & -1 \\ -1 & 1 & 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 2 & 1 & -1 \\ -1 & -2 & -1 & 1 & 2 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \end{bmatrix} =: G$$

Schur (Hadamard) Multiplication

$$G \circ G \;=$$

$$\frac{1}{4}\begin{bmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & 2 & 1 & -1 & -2 & -1 \\ -1 & 1 & 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 2 & 1 & -1 \\ -1 & -2 & -1 & 1 & 2 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \end{bmatrix} \circ \begin{bmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & 2 & 1 & -1 & -2 & -1 \\ -1 & 1 & 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 2 & 1 & -1 \\ -1 & -2 & -1 & 1 & 2 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \end{bmatrix}$$

$$G^{\circ 2} = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} & 1 \end{bmatrix}$$

Schur Multiplication Again

$$G \circ G^{\circ 2} =$$

$$\frac{1}{2}\begin{bmatrix} 2 & 1 & -1 & -2 & -1 & 1 \\ 1 & 2 & 1 & -1 & -2 & -1 \\ -1 & 1 & 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 2 & 1 & -1 \\ -1 & -2 & -1 & 1 & 2 & 1 \\ 1 & -1 & -2 & -1 & 1 & 2 \end{bmatrix} \circ \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} & 1 \\ 1 & \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 1 & \frac{1}{4} & \frac{1}{4} & 1 \end{bmatrix}$$

$$G^{\circ 3} = \begin{bmatrix} 1 & \frac{1}{8} & -\frac{1}{8} & -1 & -\frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & 1 & \frac{1}{8} & -\frac{1}{8} & -1 & -\frac{1}{8} \\ -\frac{1}{8} & \frac{1}{8} & 1 & \frac{1}{8} & -\frac{1}{8} & -1 \\ -1 & -\frac{1}{8} & \frac{1}{8} & 1 & \frac{1}{8} & -\frac{1}{8} \\ -\frac{1}{8} & -1 & -\frac{1}{8} & \frac{1}{8} & 1 & \frac{1}{8} \\ \frac{1}{8} & -\frac{1}{8} & -1 & -\frac{1}{8} & \frac{1}{8} & 1 \end{bmatrix}$$

Entrywise Powers of G Span a Vector Space

Consider the vector space $\mathcal{A}$ spanned by

$$\{J, G, G^{\circ 2}, G^{\circ 3}, G^{\circ 4}, \ldots\}$$

where the all-ones matrix J is $G^{\circ 0}$ and $G = G^{\circ 1}$.
Clearly, in our case, this space has dimension four and admits a basis of 01-matrices.

Symmetric Association Scheme

Let us say that the set X determines an *association scheme* if this vector space $\mathcal{A}$ is closed under matrix multiplication.
Observe:

- $\mathcal{A}$ is closed under Schur multiplication;
- $\mathcal{A}$ contains the identity, J, for Schur multiplication;
- $\mathcal{A}$ is closed under ordinary multiplication;
- Since the points in X are distinct, $\mathcal{A}$ contains the identity, I, for ordinary multiplication;
- Since the matrices in $\mathcal{A}$ are all symmetric, they commute.

Bose-Mesner Algebra

The vector space/ring/ring of matrices $\mathcal{A}$ is called the *Bose-Mesner algebra*. This is equivalent to a symmetric association scheme.
We may always construct two canonical bases:

$$\{A_0 = I, A_1, \ldots, A_d\}$$

(01-matrices which sum to J (pairwise disjoint support));

$$\{E_0 = \frac{1}{v} J, E_1, \ldots, E_d\}$$

(pairwise orthogonal idempotents summing to I).

Cometric (Q-polynomial) Association Scheme

Let us say that the association scheme $(X, \{A_i\}_{i=0}^{d})$ is *cometric with respect to* X if

- for each k, the vector space

$$\left\{ J, G, G^{\circ 2}, \ldots, G^{\circ k} \right\}$$

  is closed under multiplication.

Observe: Eigenvalues of G must be 0 and v/m, assuming X spans $\mathbb{R}^m$. Then we can take $E_1 = \frac{m}{v} G$,

$$E_2 = \omega_2 (G \circ G) + \omega_1 G + \omega_0 J$$

and $E_j = q_j \circ (E_1)$ where q_j is a polynomial of degree exactly j $(0 \leq j \leq d)$
(Notation: $f \circ (M)$ is matrix obtained by applying f to each entry.)

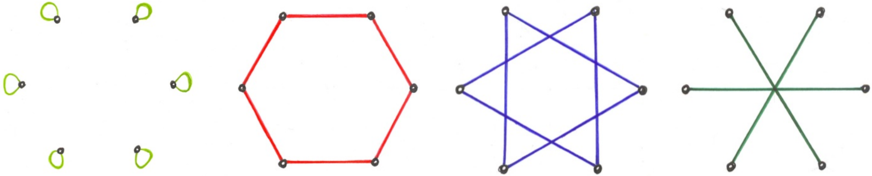
Back to the Example

For the hexagon, we obtain

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Symmetric 01-Matrices are Graphs



Back to the Example

For the hexagon, we obtain

$$E_0 = \frac{1}{6} J, \qquad E_1 = \frac{1}{3} G,$$

$$E_2 = \frac{1}{6}(3A_0 + 3A_3 - J), \qquad E_3 = \frac{1}{6}(A_0 - A_1 + A_2 - A_3)$$

Another Example: E_8 Root Lattice

- even unimodular lattice in $\mathbb{R}^8$
- kissing number 240 (optimal)
- can be identified with the integral Cayley numbers

We will focus on the spherical code consisting of the 240 (scaled) shortest vectors.

Shortest vectors

The 240 norm $\sqrt{8}$ vectors:

- $(0^6, \pm 2)$ – any two positions, all possible signs ($4 \cdot 28 = 112$ vectors)
- $(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$ – even number of minus signs ($2^7 = 128$ vectors)

Scale these to unit vectors to get $X \subset S^7$.

Among these vectors, there are only 4 non-zero angles. This gives us a 4-class cometric association scheme.

Orthogonality relations

$$A_i = \sum_{j=0}^{d} P_{ji} E_j \qquad E_j = \frac{1}{v} \sum_{i=0}^{d} Q_{ij} A_i$$

The change-of-basis matrices P and Q are called the "first and second eigenmatrices" of the scheme. A scaled version of P is called the "character table":

$$PQ = vI$$

$$MP = Q^{\top} K$$

where M is a diagonal matrix of multiplicities $m_j = \mathrm{rank}\, E_j$ and K is a diagonal matrix of valencies $v_i = \text{rowsum} A_i$.

A taste of duality

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k \qquad E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{d} q_{ij}^k E_k$$

$$A_i \circ A_j = \delta_{ij} A_i \qquad E_i E_j = \delta_{ij} E_i$$

$$A_i E_j = P_{ji} E_j \qquad A_i \circ E_j = \frac{1}{v} Q_{ij} A_i$$

$$\sum_{i=0}^{d} A_i = J \qquad \sum_{j=0}^{d} E_j = I$$

$$A_0 = I \qquad E_0 = \frac{1}{v} J$$

Metric and Cometric Schemes

Philippe Delsarte 

The scheme is *metric* (or *P-polynomial*) if there is an ordering of the A_i for which

- $p_{ij}^{k} = 0$ whenever $k > i + j$
- $p_{ij}^{i+j} > 0$ whenever $i + j \leq d$

The scheme is *cometric* (or *Q-polynomial*) if there is an ordering of the E_j for which

- $q_{ij}^{k} = 0$ whenever $k > i + j$
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Main Results

- **Delsarte**: initial list of equivalences
- **Terwilliger**: balanced set condition (and much more in P-poly case)
- **Suzuki** (1998): Essentially, there can be at most two Q-polynomial orderings
- **Suzuki** (1998): Essentially, the imprimitive ones are either Q-bipartite ("projective") or Q-antipodal ("linked")
- **Muzychuk, Williford and WJM**: Q-antipodal schemes can always be dismantled
- **Williford and WJM**: For any fixed $m_1 > 2$, there are only finitely many cometric schemes

Bannai-Ito Conjectures

Conjecture (Bannai & Ito)

Every primitive cometric scheme of sufficiently large diameter d is metric as well.

Perhaps easier?:
Order relations "naturally" so that $m_1 > Q_{11} > \cdots > Q_{d1}$.
Does A_1 have $d+1$ distinct eigenvalues?
Is there some constant $\delta \geq 1$ such that $p^k_{1j} = 0$ whenever $|k-j| > \delta$?

The Conjectures of Bannai and Ito

Let $V_j = \mathrm{colsp} E_j$ denote the j^{th} eigenspace of the cometric scheme.

Conjecture (Bannai & Ito)

The multiplicities $m_0, m_1, \ldots, m_d$ of a cometric association scheme, given by $m_j = \dim V_j$ form a unimodal sequence:

$$m_0 < m_1 \leq m_2 \leq \cdots \leq m_r \geq m_{r+1} \geq \cdots \geq m_d.$$

Conjecture (D. Stanton)

For $j < d/2$,

$$m_j \leq m_{j+1}, \qquad m_j \leq m_{d-j}.$$

Theorem (Caughman & Sagan, 2001)

If $(X, \mathcal{R})$ is also dual thin, *then Stanton's conjecture holds.*

A Source of Examples: Spherical Designs

Spherical t-Design: Finite subset $X \subset S^{m-1}$ for which

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{\|S^{m-1}\|} \int f(x) dx$$

for all polynomials f in m variables of total degree at most t.
Example: The 196,560 shortest vectors of the Leech lattice form a spherical 11-design in $\mathbb{R}^{24}$.
Seymour and Zaslavsky (1984): Such finite sets X exist for all t in each dimension m.

Cometric schemes from spherical designs

Theorem (Delsarte,Goethals,Seidel (1977))

The number s of non-zero angles in a spherical t-design is at least $t/2$. If $t \geq 2s-2$, then X carries a cometric association scheme.

Examples: 24-cell ($t = 5$, $s = 4$); E_6 ($t = 5$, $s = 4$); E_8 ($t = 7$, $s = 4$); Leech ($t = 11$, $s = 6$).

Cometric schemes from combinatorial designs

Defn: A *Delsarte t-design* in a cometric scheme (X, A) is any non-trivial subset Y of X whose characteristic vector χ_Y is orthogonal to $V_1, \ldots, V_t$.

Examples: orthogonal arrays ("dual codes"), block designs.

Theorem (Delsarte (1973))

If s non-zero relations occur among pairs of elements of Y, then $t \leq 2s$. If $t \geq 2s - 2$, then Y carries a cometric association scheme.

Cometric schemes from semilattices

Defn: The *dual width* w^* of $Y \subseteq X$ is the maximum j in the Q-polynomial ordering for which $E_j \chi_Y \neq 0$.

Theorem (Brouwer, Godsil, Koolen, WJM (2003))

For any Y in a d-class cometric scheme, $w^* \geq d - s$. If equality holds, then Y carries a cometric association scheme.

Group schemes

Every finite group G yields an association scheme via the center of the group algebra of its right regular representation $g \mapsto R_g$.
Conjugacy classes: $\mathcal{C}_0 = \{e\}, \mathcal{C}_1, \ldots, \mathcal{C}_n$

$$A_i = \sum_{g \in \mathcal{C}_i} R_g$$

Extended conjugacy classes: $\mathcal{C}'_0 = \{e\}, \mathcal{C}'_i = \mathcal{C}_i \cup (\mathcal{C}_i)^{-1}$
Symmetrized scheme:

$$A_i = \sum_{g \in \mathcal{C}'_i} R_g$$

Cometric group schemes

Theorem (Kiyota and Suzuki (2000))

The symmetrized group scheme is cometric if and only if G is one of the following groups:

- $\mathbb{Z}_n$
- S_3
- A_4
- $SL(2,3)$
- $F_{21} = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$

A Census

The following cometric association schemes are known:

- Q-polynomial distance-regular graphs (i.e., metric and cometric)
- duals of metric translation schemes
- bipartite doubles of Hermitian forms dual polar spaces $[{}^2A_{2d-1}(r)]$ (Bannai & Ito)
- schemes arising from linked systems of symmetric designs (3-class, Q-antipodal) [Cameron & Seidel]
- extended Q-bipartite doubles of linked systems (4-class, Q-bipartite and Q-antipodal) [Muzychuk, Williford, WJM]
- real MUBS [Bannai & Bannai, LeCompte & Owens & WJM]

Census

census of cometric schemes, continued:

- the block schemes of the Witt designs 4-(11,5,1), 5-(24,8,1) and a 4-(47,11,8) design (Delsarte) (primitive 3-class schemes on 66, 759 and 4324 vertices resp.)
- the block schemes of the 5-(12,6,1) design and the 5-(24,12,48) design (Q-bipartite 4-class schemes on 132 and 2576 vertices, resp.)
- shortest vectors in lattices E_6, E_7, E_8 (4-class, Q-bipartite)
- the scheme on the vertices of the 24-cell (4-class, Q-bipartite, Q-antipodal, 24 vertices)

Census

census of cometric schemes, continued:

- the scheme on the shortest vectors in the Leech lattice (6-class, Q-bipartite, 196560 vertices)
- 5 schemes arising from derived designs of this:

3-class	2025 vertices	primitive
4-class	2816	Q-bipartite
4-class	4600	Q-bipartite
4-class	7128	primitive
5-class	47104	primitive

- Q-bipartite quotient of Leech lattice example (3-class, primitive)
- three more schemes arising from lattices (4-, 5-, 11-class, Q-bipartite)

Census

census of cometric schemes, continued:

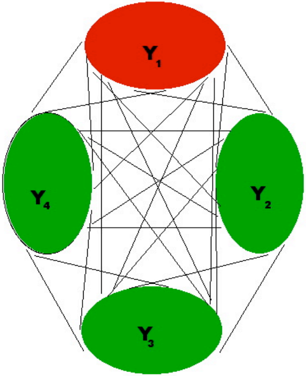
- three schemes from dismantling dual schemes of metric translation schemes (4-, 5-, and 6-class, all Q-antipodal)
- One infinite family ("triality") and three exceptional Q-antipodal schemes with 4 classes [D.G. Higman]
- One infinite family from hemisystems in generalized quadrangles (4-class, Q-antip.) [Cossidente & Penttila]
- One very new infinite family (3-class, primitive) [Penttila & Williford]

Dismantlability

Theorem (Muzychuk, Williford, WJM (2007))

Every Q-antipodal scheme is dismantlable: the subscheme induced on any non-trivial collection of w' *Q-antipodal classes is cometric for* $w' \geq 1$ *and Q-antipodal with d classes for* $w' > 1$.

Dismantlability



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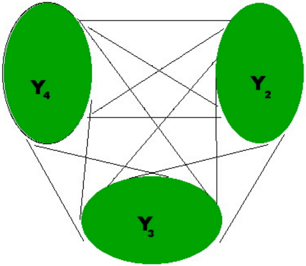

Trivial cases

- halved graph of a bipartite Q-polynomial distance-regular graph
- linked systems of symmetric designs (by defn.)

A new example via dismantling

Coset graph of the shortened ternary Golay code:

- intersection array $\{20, 18, 4, 1; 1, 2, 18, 20\}$

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- Remove one of these to obtain a Q-antipodal scheme on 162 vertices having $w = 2$ Q-antipodal classes which is not metric
- parameters

$$d = 4, \ v = 162, \ \iota^*(X, \mathbf{A}) = \{20, 18, 3, 1; 1, 3, 18, 20\}$$

formally dual to those of an unknown diameter four bipartite distance-regular graph.

Dismantling the dual of a coset graph

- Two more distance-regular coset graphs yield Q-antipodal schemes with five and six classes.

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- Parameters
  $d = 5, \; v = 486,$
  $$\iota^*(X, \mathbf{A}) = \{22, 20, \frac{27}{2}, 2, 1; \; 1, 2, \frac{27}{2}, 20, 22\}, \; w = 2$$
  $d = 6, \; v = 1536,$
  $$\iota^*(X, \mathbf{A}) = \{21, 20, 16, 8, 2, 1; \; 1, 2, 4, 16, 20, 21\}, \; w = 3.$$

Dismantling the dual of a coset graph

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  $d = 6, \ v = 1536,$
  $$\iota^*(X, \mathbf{A}) = \{21, 20, 16, 8, 2, 1;\ 1, 2, 4, 16, 20, 21\}, \ w = 3.$$
- This last scheme is formally dual to a distance-regular graph which was proven not to exist by Brouwer, Cohen and Neumaier.

The 4-cycle

$$E_1 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Ring homomorphism $\gamma : \mathbb{C}[Z_1, Z_2, Z_3, Z_4] \to \mathbb{C}^4$ takes

$$Z_1 \mapsto \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \qquad Z_2 \mapsto \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \text{etc.}$$

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Ring homomorphism $\gamma : \mathbb{C}[Z_1, Z_2, Z_3, Z_4] \to \mathbb{C}^4$ takes

$$4Z_1+2Z_2 \mapsto \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \qquad Z_1Z_2 \mapsto 0, \quad Z_1Z_4 \mapsto \frac{1}{4}\begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \text{etc.}$$

An elementary ring homomorphism

In general, let $(X, \mathcal{R})$ be a cometric association scheme on v vertices with first primitive idempotent E_1.
Let $\gamma : \mathbb{C}[Z_1, \ldots, Z_v] \to \mathbb{C}^X$ via

$$Z_a \mapsto \bar{a}$$

(the a-column of E_1) and extending linearly and via the Schur product $\circ$.
E.g., $Z_a Z_b^2 - 3Z_a \mapsto (\bar{a} \circ \bar{b} \circ \bar{b}) - 3\bar{a}$
We are interested in $\mathcal{I} = \ker \gamma$.

The Q-Ideal

Object of study: $\mathcal{I} = \ker \gamma$

Theorem

$\mathcal{I}$ is the set of polynomials in $\mathbb{C}[Z_1, \ldots, Z_v]$ which vanish on each column of E_1

Here, $v = |X|$ is the number of vertices in the cometric scheme $(X, \mathcal{R})$. Equivalently, we can look at an ideal $\mathcal{I}_N$ in the ring $\mathbb{C}[Y_1, \ldots, Y_{m_1}]$.

The Q-Ideal

Observe: The columns of E_1, and hence the entire association scheme and its parameters, can be recovered from $\mathcal{I}$

Observe: The automorphism group of the association scheme acts on the polynomial ring preserving the ideal $\mathcal{I}$.

Some Motivation

- **Delsarte, Goethals, Seidel:** If $u \in V_i$ and $v \in V_j$ and $q_{ij}^k = 0$, then $u \circ v \perp V_k$.

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$$c(u \circ v \circ w) - d(v \circ v)$$

  and we want to know when two of these are equal.
- Nice designs and codes can be efficiently encoded as polynomials. E.g.
- Fano plane $\mathcal{D}$ in $J(7,3)$ yields subideal of $\mathcal{I}$ consisting of those polynomials involving only $\{Z_a | a \in \mathcal{D}\}$

Very small degree

Object of study: $\mathcal{I} = \ker \gamma$

- $\mathcal{I}$ contains $v - m_1$ linearly independent linear polynomials, spanning the nullspace of E_1
- $\mathcal{I}$ contains all multiples of

$$Z_1^2 + Z_2^2 + \cdots + Z_v^2 - \frac{m_1}{v} =: \| \cdot \|^2 - c$$

Small Degree Generators

In the n-cube, the code $C = \{ a \mid a_1 = 0 \}$ has width $n-1$ and dual width $w^* = 1$. (I.e., $E_j x_C = 0$ for all $j > w^*$.)

This gives a quadratic polynomial in our ideal:

$$F = \left(\sum_{c \in C} Z_c - \frac{1}{2} \right) \left(\sum_{c \in C} Z_c + \frac{1}{2} \right)$$

As C ranges over the dim. $n-1$ subcubes, this gives a set of quadratic polynomials which generate $\mathcal{I}_N$.

Small Degree Generators

The ideal $\mathcal{I}$ is generated by linear and quadratic polynomials for the following classical families of association schemes:

- Hamming schemes $H(n, q)$
- Johnson schemes $J(n, k)$
- Grassman schemes $G_q(n, k)$
- bilinear forms schemes $B_q(m, n)$

Proof: There are enough subsets of dual width one that each vertex is uniquely determined by those such subsets which contain it.

More Small Degree Generators

- 24-cell: I generated by polys. of degree at most four
- E_6: " degree at most three
- E_7: " degree at most four
- E_8: " degree at most four
- Leech lattice: will require polynomials of degree six, at least.

Spherical t-Designs

Recall: A subset X of the unit sphere S^{m-1} is a *spherical t-design* if, for every polynomial F in m variables, the average of F over X is the same as the average of F over the sphere.

Spherical t-Designs

Observe: If X is a spherical $2s$-design and F is a polynomial in $\mathcal{I}$ of degree $\leq s$, then F is a multiple of $\|\cdot\|^2 - c$.

Proof: F^2 is strictly positive and zero at every point of X. Since its degree is $\leq 2s$, it must be zero on the entire sphere.

Some Spherical t-Designs

- 24-cell: $m = 4$, $|X| = 24$, $t = 5$
- E_6: $m = 6$, $|X| = 72$, $t = 5$
- E_7: $m = 7$, $|X| = 126$, $t = 5$
- E_8: $m = 8$, $|X| = 240$, $t = 7$ (tight)
- Leech lattice: $m = 24$, $|X| = 196560$, $t = 11$

How fast can m_j grow?

We can now view the j^{th} eigenspace of the association scheme as the space of polynomials of degree j on X.
The multiplicity m_j is the dimension of this space.

Absolute Bound: $\sum_{k:q_{ij}^k>0} m_k \le m_i m_j$
gives

$$m_2 \le \binom{m+1}{2} - 1$$

Equality holds iff $q_{11}^1 = 0$ and $q_{11}^2 = \frac{2m}{m+2}$.

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Equality holds iff $q_{11}^1 = 0$ and $q_{11}^2 = \frac{2m}{m+2}$.

This occurs for the 24-cell, E_6, E_7, E_8, the Leech lattice and several of its derived designs.

What is the dual concept to a Moore graph?

Eiichi Bannai determined that the dual object of a Moore graph is a tight spherical t-design. So the only examples are polygons, the icosahedron,

- (min length vectors of the) Leech (lattice)
- a derived spherical design of this on 4600 points
- E_8
- two derived designs of E_8
- a system of 276 equiangular lines in $\mathbb{R}^{23}$ arising from $Co._3$
- a strongly regular graph on 275 vertices related to this one

Any other tight spherical t-design must have $t \in \{4, 5, 7\}$ and special parameters.

Imprimitive Q-polynomial Schemes

If the scheme is Q-bipartite, then $-X = X$. So, eliminating $\| \cdot \|^2 - c$, $\mathcal{I}$ can be expressed as a homogeneous ideal.

If the scheme is Q-antipodal with ideal $\mathcal{I}$ and some Q-antipodal subobject (via dismantling) has ideal $\mathcal{J}$, then $\mathcal{I} \subseteq \mathcal{J}$.

(Can this help us extend known Q-antipodal schemes?)

Homotopy

Let Γ be a distance-regular graph (metric association scheme) and let x be any vertex. Equivalence classes of closed walks in Γ beginning and ending at x form a group under concatenation and reversal.

This is the *fundamental group* $\pi(\Gamma, x)$ of Γ and essentially does not depend on x.

A Sequence of Homotopy Groups

H. Lewis (2000): 

The *essential length* of a walk w of the form pqp^{-1} is at most the length of walk q.

Definition: Let $\pi(\Gamma, x, k)$ be the subgroup of $\pi(\Gamma, x)$ generated by equivalence classes of closed walks of essential length at most k.

Theorem (Lewis)

If Γ is a distance-regular graph of diameter d, then

$$\{e\} = \pi(\Gamma, x, 0) = \pi(\Gamma, x, 1) = \pi(\Gamma, x, 2) \subseteq \cdots$$

$$\subseteq \pi(\Gamma, x, 2d+1) = \pi(\Gamma, x).$$

Translation Schemes

A translation scheme is a scheme $(X, \mathcal{R})$ where X is a finite abelian group and $(a, b) \in R_i$ implies $(a + c, b + c) \in R_i$.

We assume $(X, \mathcal{R})$ is a cometric translation scheme and then there is a distance-regular graph Γ defined on the group $X^\dagger$ of characters of X.

Some set S_1 of characters forms a basis for the first eigenspace in the Q-polynomial ordering of $(X, \mathcal{R})$. The graph has edges $(\psi, \psi \circ \chi)$ for $\chi \in S_1$.

So if $S_1 = \{\chi_1, \ldots, \chi_m\}$, then each walk $w = \psi_0, \psi_1, \ldots$ in Γ can be described by giving its starting point ψ_0, together with a sequence $h_1, h_2, \ldots, h_s$ for which $\psi_j = \psi_{j-1} \circ \chi_{h_j}$.

Homotopy and Duality

In a cometric translation scheme, each closed walk in the dual distance-regular graph Γ yields a polynomial in $\mathcal{I}_N$ and these generate $\mathcal{I}_N$:

$$F_w = Y_{h_1} Y_{h_2} \cdots Y_{h_s} - 1$$

So if Lewis's subgroup $\pi(\Gamma, x, k)$ is the entire fundamental group $\pi(\Gamma, x)$, then the ideal $\mathcal{I}_N$ is generated by polynomials of total degree at most $(k+1)/2$.

Cycles are special

Here is a Gröbner basis for the ideal $\mathcal{I}_N$ (dimension two) in the case of the n-cycle:

$$X^2 + Y^2 - 1, \qquad (X-1)(X-\zeta_1)\cdots(X-\zeta_{\lfloor n/2 \rfloor})$$

where (with $\alpha = \frac{2\pi}{n}$) we have $\zeta_k = \cos(k\alpha)$.

The Q-Ideal Conjecture

Conjecture

There is a universal constant K such that, for any cometric association scheme with $m_1 > 2$, the ideal $\mathcal{I}$ is generated by polynomials of total degree at most K.

A Partial Result

Theorem (Williford & WJM, 2009)

For each integer $m > 2$, there is an integer $K(m)$ such that, for any cometric association scheme with $\mathrm{rank}\, E_1 = m$, the ideal I is generated by polynomials of total degree at most $K(m)$.

Remark: We really proved simply that, for $m > 2$, there can be only finitely many cometric association schemes with $m_1 = m$.

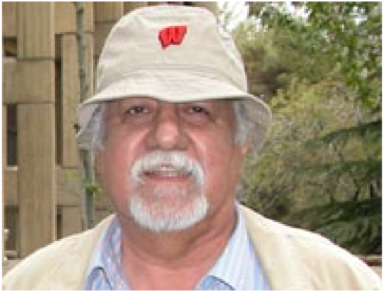
Consequences

We saw that there exist spherical t-designs for all t.
If this universal bound K exists, then no spherical t-design with $t > 2K$ can give a cometric association scheme (except polygons).

The End

Thank you all.

Happy Birthday Reza!



Happy Birthday IPM.