Designs in Product Association Schemes

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Abstract

Let $(Y, \mathcal{A})$ be an association scheme with primitive idempotents $E_0, E_1, \ldots, E_d$. For $\mathcal{T} \subseteq \{1, \ldots, d\}$, a Delsarte $\mathcal{T}$-design *in $(Y, \mathcal{A})$ is a subset D of Y whose characteristic vector is annihilated by the idempotents E_j $(j \in \mathcal{T})$. The case most studied is that in which $(Y, \mathcal{A})$ is Q-polynomial and $\mathcal{T} = \{1, \ldots, t\}$. For many such examples, a combinatorial characterization is known, giving an equivalence between Delsarte $\mathcal{T}$-designs and poset t-designs in what we call here "Q-posets". For example, combinatorial t-designs (i.e., block designs) can be described via the truncated Boolean lattice while orthogonal arrays can be described via the Hamming lattice.*

For $1 \leq i \leq m$, let $(Y_i, \mathcal{A}_i)$ be a Q-polynomial association scheme. Assume that Delsarte t-designs in each $(Y_i, \mathcal{A}_i)$ are characterised as poset t-designs in a Q-poset $\mathcal{P}_i$ attached to that scheme. With these assumptions, we consider the product association scheme $(\times Y_i, \mathcal{A})$. The primitive idempotents for this scheme naturally inherit the partial order structure of a product of chains. Our main result, Theorem 2.3, characterises Delsarte $\mathcal{T}^*$-designs in $(\times Y_i, \mathcal{A})$ as poset designs in the product poset $\times \mathcal{P}_i$ where $\mathcal{T}$ is any downset in the product of chains. Using this characterisation, we immediately obtain linear programming bounds for a wide variety of combinatorial objects. On the other hand, if we assume each component scheme is Q-polynomial, we obtain bounds on the size and degree of such a $\mathcal{T}$-design analogous to Delsarte's bounds for t-designs in Q-polynomial association schemes.

1 Overview and Background

We are interested in applications of the theory of association schemes to problems in coding theory and design theory. This investigation is motivated by two applications. In [10],

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Levenshtein introduces split orthogonal arrays (see Example 2.4 below) and applies the theory of Krawtchouk polynomials to obtain bounds on the size of such objects. It was clear to the author early on that similar results hold for mixed-level orthogonal arrays. Independently, Sloane and Stufken [17] derived a linear programming bound for these objects. Our approach shows how the two studies can be carried out simultaneously and elegantly if one considers the product of Hamming association schemes and the associated product of Hamming lattices.

In fact, much more can be accomplished if one abstracts the essential features of the relationship between the Hamming association scheme and (incidence matrices of) the Hamming lattice. With this motivation, we introduce in Section 2.1 the concept of a *Q-poset* attached to an association scheme. (While the definition can be applied to arbitrary association schemes, the case where the scheme is Q-polynomial is our focus.) The main result of the paper is Theorem 2.3 in Section 2.2 which uses these partially ordered sets to characterise Delsarte $\mathcal{T}^*$-designs in a product association scheme where each component scheme is assumed to have an attached Q-poset and where $\mathcal{T}$ is a downset in a product of chains.

The balance of the paper explores bounds on $\mathcal{T}$-designs in product schemes where each component is Q-polynomial. The general results include: a linear programming bound (which falls out immediately from Theorem 2.3 using the standard theory); a Delsarte bound (Theorem 3.2) analogous to the Rao bound for orthogonal arrays and the Ray-Chaudhuri/Wilson bound for combinatorial t-designs; a degree bound (Theorem 3.3). Numerous examples are included to demonstrate the power and wide applicability of these bounds.

1.1 Association Schemes

All material in this section is standard (see [4, 1, 2, 8]).

Let X be a finite set of cardinality $v > 0$ and let $\mathcal{A} = \{A_0, A_1, \ldots, A_d\}$ be a set of $v \times v$ symmetric 01-matrices with rows and columns indexed by X and with linear span $\mathbb{A} \leq \mathbb{R}^{v\times v}$. We say that $(X, \mathcal{A})$ is a (symmetric) *association scheme* if $\mathcal{A}$ satisfies

(i) $A_0 = I$;

(ii) $\sum_{i=0}^{d} A_i = J$ (the all-ones matrix);

(iii) for $0 \leq i, j \leq d$, $A_iA_j \in \mathbb{A}$.

To each matrix A_i ($i \neq 0$) is associated its graph G_i with vertex set X: A_i is the adjacency matrix of G_i. The above definition is customarily given in the language of relations. We will use graph and matrix terminology interchangeably. The vector space $\mathbb{A}$ spanned by the matrices in $\mathcal{A}$ is, by condition **(iii)**, an algebra; this is the *Bose-Mesner algebra* of the association scheme. Extensive background material on association schemes can be found in the references.

A. The Hamming Scheme

In terms of applications, two important association schemes are the Hamming scheme and the Johnson scheme. The *Hamming scheme* $H(n,q)$ has as its vertices the set X of all words of length n over an alphabet $\mathcal{Q}$ having q elements. The graphs $G_1,\ldots,G_n$ are defined by Hamming distance: if $x,y\in X$, then $(x,y)\in G_k$ where k is the number of positions in which the words x and y differ. It is well-known that the mapping $(x,y)\mapsto k$ is a metric. So the entire association scheme is determined by the *Hamming graph* G_1. Thus the Hamming scheme is a *metric* (hence, *P-polynomial*) association scheme. (Metric association schemes are the same as distance-regular graphs.) As examples, for $q=2$, G_1 is the n-cube and for $n=2$, G_1 is the grid graph $K_q\times K_q$.

Several authors have associated to this scheme a partially ordered set $(\mathcal{P},\preceq)$, commonly called the *Hamming lattice* (see [5, 19, 14, 8]). Adjoin to the alphabet $\mathcal{Q}$ a new symbol "$\cdot$". The elements of $\mathcal{P}$ are the words of length n over $\mathcal{Q}\cup\{\cdot\}$. For $x,y\in\mathcal{P}$, we have $x\preceq y$ if and only if, for all i, $y_i=\cdot$ implies $x_i=\cdot$. This poset has rank function ℓ given by $\ell(x)=|\{i: x_i\neq\cdot\}|$. So the elements of X are precisely the members of $\mathcal{P}$ having maximum rank n. It is immediately evident that a subset $D\subset X$ is an orthogonal array of strength t in $H(n,q)$ if and only if, in $(\mathcal{P},\preceq)$, there exists λ such that every element of rank t is dominated by exactly λ members of D.

B. The Johnson Scheme

Let $\mathcal{V}$ be a set of n points and let k satisfy $1\leq k\leq n/2$. The *Johnson scheme* $J(n,k)$ has vertex set X consisting of all k-element subsets of $\mathcal{V}$ and $(x,y)\in G_i$ $(1\leq i\leq k)$ if $|x\cap y|=k-i$. Again, we have a metric association scheme; G_1 is called the Johnson graph. To this association scheme, we attach the truncated boolean lattice $(\mathcal{P},\subseteq)$ where $\mathcal{P}=\{S\subseteq\mathcal{V}: |S|\leq k\}$ with rank function $\ell(S)=|S|$. A combinatorial t-design is simply any subset D of X having the property that every element S of rank t is dominated by exactly λ members of D for some fixed λ.

The Johnson graphs and Hamming graphs are examples of distance-regular graphs and these two families are well-studied. In particular, Delsarte [5] analysed the relationship between these schemes and the above-mentioned posets (as well as their q-analogues) in the language of regular semilattices.

1.2 Linear Programming

Let $(X,\mathcal{A})$ be an association scheme with adjacency matrices $\mathcal{A}=\{A_0,\ldots,A_d\}$ and Bose-Mesner algebra $\mathbb{A}$. Since the A_i are 01-matrices with pairwise disjoint supports, this algebra is also closed under *Schur (entrywise) multiplication*, denoted $\circ$. The matrices A_i form a basis of mutually orthogonal idempotents for $\mathbb{A}$ with respect to this multiplication ($A_i\circ A_j=\delta_{i,j}A_i$). There is also a unique basis $\{E_0,E_1,\ldots,E_d\}$ of mutually orthogonal idempotents with respect to ordinary matrix multiplication (which we call the "primitive idempotents").

The change-of-basis matrices P and Q, defined by

$$A_i = \sum_{j=0}^{d} P_{ji} E_j \qquad \text{and} \qquad E_j = \frac{1}{v} \sum_{i=0}^{d} Q_{ij} A_i \tag{1}$$

satisfy the orthogonality relations

$$PQ = vI \qquad \text{and} \qquad MP = Q^T K \tag{2}$$

where K is a diagonal matrix with K_{ii} equal to the valency of (the regular graph) G_i and M is a diagonal matrix with $M_{jj} = f_j := \mathsf{rank}\, E_j$. We refer to these simply as the "P-matrix" and "Q-matrix" of the scheme.

For the convenience of the reader mainly interested in applications, we record here the change-of-basis matrices for the two examples described above. For the Hamming scheme $H(n,q)$, we have $P_{ji} = K_i(j)$ where

$$K_i(x) = \sum_{\ell=0}^{i} (-1)^\ell (q-1)^{i-\ell} \binom{x}{\ell} \binom{n-x}{i-\ell} \tag{3}$$

is the *Krawtchouk polynomial* familiar to coding theorists and we have $Q = P$. (This is a self-dual association scheme.) For the Johnson scheme $J(n,k)$, we have $P_{ji} = E_i(j)$ where

$$E_i(x) = \sum_{\ell=0}^{i} (-1)^\ell \binom{x}{\ell} \binom{k-x}{i-\ell} \binom{n-k-x}{i-\ell} \tag{4}$$

is the *dual Hahn polynomial* of degree i. The rank of E_j in this case is given by $f_j = \binom{n}{j} - \binom{n}{j-1}$, the valency of G_i is given by $k_i = \binom{k}{i}\binom{n-k}{i}$, and Q is obtained from P using the orthogonality relations above.

Given any subset D of the vertices of an association scheme $(X, \mathcal{A})$ on v points, we build the characteristic vector of D, denoted χ_D, which is a 01-vector of length v with y-entry equal to 1 if and only if $y \in D$. We are also interested in the *inner distribution* vector of D: $\mathbf{a} = [a_0, a_1, \ldots, a_d]$ is defined by

$$a_i = \frac{1}{|D|} \chi_D^T A_i \chi_D \tag{5}$$

which, in the case of a distance-regular graph, gives the average number of elements of D at distance i from any element of D. For linear codes, $\mathbf{a}$ is simply the list of coefficients of the weight enumerator. For combinatorial t-designs, a_i measures the average number of blocks meeting a given block in $k - i$ points.

Consider the product $\mathbf{a}Q$, which is called the *MacWilliams transform* of $\mathbf{a}$. By Equation (1b), we have

$$(\mathbf{a}Q)_j = \frac{v}{|D|} \chi_D^T E_j \chi_D$$

and, since $E_j^2 = E_j$, this quantity is always non-negative (see also [2, Sec. 2.5]):

Theorem 1.1 (Delsarte's Linear Programming Bound [4, Thm. 3.3]) *For any subset D of the vertices X of an association scheme, the MacWilliams transform of the inner distribution vector is non-negative:* $\mathbf{a}Q \geq 0$. $\square$

For $\mathcal{T} \subseteq \{1, \ldots d\}$, a *Delsarte $\mathcal{T}$-design* is a subset $D \subseteq X$ satisfying $E_j \chi_D = 0$ for $j \in \mathcal{T}$. This is clearly equivalent to the condition $(\mathbf{a}Q)_j = 0$ for all $j \in \mathcal{T}$. In the case where $\mathcal{T} = \{1, \ldots, t\}$, we will simply say D is a *Delsarte t-design*.

2 Combinatorial characterizations of Delsarte $\mathcal{T}$-designs

2.1 Q-posets

Let $(\mathcal{P}, \preceq)$ be a partially ordered set (*poset*). If $x \preceq y$, we say y *dominates* x. If $x \prec y$ and there is no z with $x \prec z \prec y$, we say y *covers* x. For a non-negative integer d, write $[d] = \{0, 1, \ldots, d\}$. A *rank function* on $(\mathcal{P}, \preceq)$ is a surjection $\ell : \mathcal{P} \to [d]$ satisfying the following condition: *if $\ell(x) = i$ and y covers x, then $\ell(y) = i + 1$.* A *ranked poset* is then simply a poset $(\mathcal{P}, \preceq)$ with a specified rank function.

If $(\mathcal{P}, \preceq)$ is a ranked poset with rank function ℓ, then $\mathcal{P}$ is partitioned into *fibres* $\mathcal{P}^0, \mathcal{P}^1, \ldots, \mathcal{P}^d$ where

$$\mathcal{P}^i = \{x \in \mathcal{P} : \ell(x) = i\}$$

consists of those objects of *rank* i. For $i \in [d]$, let W_i be the incidence matrix of the i^{th} fibre $\mathcal{P}^i$ versus the top fibre $\mathcal{P}^d$. The rows of W_i are indexed by elements of $\mathcal{P}^i$ and the columns are indexed by elements of $\mathcal{P}^d$.

As indicated above, a number of design-theoretic objects have a common definition in the language of posets. Let $(\mathcal{P}, \preceq)$ be a ranked poset as above. Extending Delsarte's definition [5], let us call $D \subseteq \mathcal{P}^d$ a *poset t-design* in $(\mathcal{P}, \preceq)$ if there exist constants $\lambda_0, \lambda_1, \ldots, \lambda_t$ such that, for $0 \leq i \leq t$ and for all $x \in \mathcal{P}^i$,

$$|\{y \in D : \ x \preceq y\}| = \lambda_i.$$

For example, a combinatorial t-design is equivalent to a poset t-design in the truncated Boolean lattice and an orthogonal array of strength t is a poset t-design in the Hamming lattice.

Let $(X, \mathcal{A})$ be a symmetric association scheme having d classes and a fixed ordering $E_0, \ldots, E_d$ on its primitive idempotents. Denote by V_j the j^{th} eigenspace, $\mathsf{rowsp}\, E_j$, of the scheme. We say a ranked poset $(\mathcal{P}, \preceq)$ is a *Q-poset* for $(X, \mathcal{A})$ (with respect to the ordering $0, 1, \ldots, d$) if the following axioms are satisfied:

(1) the top fibre $\mathcal{P}^d$ is the set X;

(2) for $0 \leq i \leq d$, W_i has constant row sum;

(3) for $0 \le i \le d$, $V_i \subseteq \mathsf{rowsp}\, W_i \subseteq V_0 \oplus V_1 \oplus \ldots \oplus V_i$.

Note: Unless explicitly stated to the contrary, the only ordering assumed on the primitive idempotents of an association scheme is the natural ordering $E_0, E_1, \ldots, E_d$. This ordering, usually dictated by the application, is to be assumed in any statement regarding attached Q-posets or the Q-polynomial property.

Remark: Viewing the rows of the incidence matrices W_i as subsets of X, our axioms **(2)** and **(3)** can be stated as conditions on certain collections $\mathcal{E}_t$ $(0 \le t \le d)$ of "antidesigns". Following Roos [16], we call a subset C of X an *antidesign with dual diameter* t if $E_j \chi_C = 0$ for $j > t$. Then our axioms amount to assuming, for each $0 \le t \le d$, the existence of a collection $\mathcal{E}_t$ of antidesigns such that the members of $\mathcal{E}_t$ are equicardinal and their characteristic vectors span a space containing V_t. See Meyerowitz [14] for a case study of the connection between antidesigns and Q-posets.

Lemma 2.1 *Let $(\mathcal{L}, \preceq)$ be a regular semilattice [5] and let $(X, \mathcal{A})$ be the association scheme carried by the top fibre. Then there exists a subposet $(\mathcal{P}, \preceq)$ of $(\mathcal{L}, \preceq)$ which is a Q-poset for $(X, \mathcal{A})$.*

Proof: See [5] for definitions of these objects. There, Delsarte proves that there exist fibres $\mathcal{L}^0, \mathcal{L}^{j_1}, \ldots, \mathcal{L}^{j_d}$ of $(\mathcal{L}, \preceq)$ which, together with the incidence matrices between them, satisfy conditions **(1)-(3)** above. □

For example, the Hamming lattice is a Q-poset for the Hamming scheme $H(n,q)$ and, for $v \ge 2k$, the truncated Boolean lattice is a Q-poset for the Johnson scheme $J(v,k)$.

For any Q-poset derived from a regular semilattice in this way, there are constants

$$\mu_{i,j} = |\{y \in \mathcal{P}^j : z \preceq y \preceq x\}|, \qquad \theta_{i,j} = |\{y \in \mathcal{P}^j : z \preceq y\}|$$

which are independent of $z \in \mathcal{P}^i$ and $x \in X$ provided $z \preceq x$. If such constants exist for a Q-poset $(\mathcal{P}, \preceq)$, let us say that $(\mathcal{P}, \preceq)$ is a *regular Q-poset* (cf. [5]). In such cases, a set D for which λ_t is well-defined is automatically a t-design in $(\mathcal{P}, \preceq)$.

The following lemma was proved in [5] for regular semilattices.

Lemma 2.2 *Let $(X, \mathcal{A})$ be a symmetric association scheme. Let $(\mathcal{P}, \preceq)$ be a Q-poset for $(X, \mathcal{A})$ with incidence matrices W_i. Then for $D \subseteq X$ with characteristic vector χ_D, the following are equivalent:*

(i) *D is a poset t-design in $(\mathcal{P}, \preceq)$;*

(ii) *there exist constants λ_i, $0 \le i \le t$ such that $W_i \chi_D = \lambda_i \mathbf{1}$;*

(iii) *for $1 \le j \le t$, $E_j \chi_D = 0$;*

(iv) *for $1 \le j \le t$, $(\mathbf{a}Q)_j = 0$.*

Proof: The equivalence of **(i)** and **(ii)** is immediate from the definitions. The equivalence of **(iii)** and **(iv)** is entirely standard [4, p27].

(ii)⇒(iii): Let $\mathbf{e}$ be any row of E_j $(1 \leq j \leq t)$. By **(3)**, we may write $\mathbf{e} = \mathbf{r}W_j$ for some vector $\mathbf{r}$ of length $|\mathcal{P}^j|$. Let ρ be the row sum of W_j (using **(2)**); then we have $W_j\mathbf{1}^T = \rho\mathbf{1}^T$. Since $j \neq 0$,

$$0 = \langle \mathbf{e}, \mathbf{1} \rangle = \mathbf{r}W_j\mathbf{1}^T = \rho\mathbf{r}\mathbf{1}^T.$$

This allows us to compute $\langle \mathbf{e}, \chi_D \rangle$. We have

$$\langle \mathbf{e}, \chi_D \rangle = \mathbf{r}W_j\chi_D = \lambda_j\mathbf{r}\mathbf{1}^T = 0.$$

(iii)⇒(ii): Let $\mathbf{w}$ be any row of W_i $(0 \leq i \leq t)$. With $\mathbf{e}_j = \mathbf{w}E_j$, we have $\mathbf{w} = \sum_{j=0}^{i} \mathbf{e}_j$ using **(3)**. Now if $\alpha = \frac{1}{|X|}\langle \mathbf{w}, \mathbf{1} \rangle$, then $\mathbf{e}_0 = \alpha\mathbf{1}$ and α is independent of the choice of $\mathbf{w}$ since W_i has constant row sum. Consequently,

$$\langle \mathbf{w}, \chi_D \rangle = \sum_{j=0}^{i} \langle \mathbf{e}_j, \chi_D \rangle = \langle \mathbf{e}_0, \chi_D \rangle = \alpha|D|$$

is independent of the choice of $\mathbf{w}$ and we have $W_i\chi_D = \lambda_i\mathbf{1}^T$ where $\lambda_i|X| = \rho|D|$ with ρ being the row sum of W_i. □

While we use the term Q-poset, the existence of such a poset attached to an association scheme $(X, \mathcal{A})$ does not seem to imply that the scheme is Q-polynomial. It remains open to decide whether there exists a non-Q-polynomial association scheme with an attached Q-poset.

Example 2.1: The Shrikhande graph (see [2, p104]) is a distance-regular graph of diameter two. This is a Cayley graph for $\mathbb{Z}_4 \times \mathbb{Z}_4$ with $(0,0)$ adjacent to $\pm(0,1)$, $\pm(1,0)$, and $\pm(1,3)$. The adjacency matrix of this graph has eigenvalues $\theta_0 = 6$, $\theta_1 = 2$, and $\theta_2 = -2$. Song [18] found several Q-posets for this association scheme. One of them is described as follows. Let $\mathcal{P}^0 = \{X\}$ and let $\mathcal{P}^2 = X$. Let $Y = \{(a,b) : b = 0, 1\}$. There are twelve subsets of X in the orbit of Y under the automorphsim group of this graph. Call these $Y_1, \ldots, Y_{12}$ and set $\mathcal{P}^1 = \{Y_1, \ldots, Y_{12}\}$. Now order $\mathcal{P} = \mathcal{P}^0 \cup \mathcal{P}^1 \cup \mathcal{P}^2$ by reverse inclusion. This is a Q-poset for the Shrikhande graph. For example, the set $D = \{(a,a) : a \in \mathbb{Z}_4\}$ is both a poset 1-design and a Delsarte 1-design for this situation.

2.2 A combinatorial characterisation of designs in product schemes

For $1 \leq i \leq m$, let $(Y_i, \mathcal{A}_i)$ be a d_i-class association scheme with adjacency matrices $\mathcal{A}_i$. The *direct product* of these schemes is the association scheme

$$(X, \mathcal{A}) = (Y_1, \mathcal{A}_1) \otimes \cdots \otimes (Y_m, \mathcal{A}_m)$$

defined by

$$X = Y_1 \times Y_2 \times \cdots \times Y_m$$

and

$$\mathcal{A} = \{\otimes_{i=1}^{m} M_i : \ M_i \in \mathcal{A}_i, \ 1 \leq i \leq m\}$$

where

$$\otimes_{i=1}^{m} M_i = M_1 \otimes M_2 \otimes \cdots \otimes M_m$$

is the m-fold Kronecker product of matrices. The fact that this always gives an association scheme follows from the properties

$$(\otimes_{i=1}^{m} M_i)(\otimes_{i=1}^{m} N_i) = \otimes_{i=1}^{m}(M_i N_i),$$

$$(\otimes_{i=1}^{m} M_i) \circ (\otimes_{i=1}^{m} N_i) = \otimes_{i=1}^{m}(M_i \circ N_i)$$

of the Kronecker product. From the first identity, it follows that the P-matrix for this product scheme is given by the Kronecker product of the P-matrices for the component schemes $(Y_i, \mathcal{A}_i)$. Similarly, the second change-of-basis matrix Q for the product scheme is given by $\otimes_{i=1}^{m} Q_i$ where Q_i is the Q-matrix for the i^{th} component scheme.

Assume that each component scheme $(Y_i, \mathcal{A}_i)$ has an attached Q-poset $(\mathcal{P}_i, \preceq_i)$. Consider the product poset $(\mathcal{P}, \trianglelefteq)$ given by

$$\mathcal{P} = \{\underline{p} = (p_1, p_2, \ldots, p_m) : \ p_i \in \mathcal{P}_i\}$$

with $\underline{p} \trianglelefteq \underline{q}$ if $p_i \preceq_i q_i$ for each i. On this poset, we have a natural vector-valued height function. If $\underline{p} = (p_1, p_2, \ldots, p_m)$ and $\ell(p_j) = i_j$ for $1 \leq j \leq m$, then we will write $\ell(\underline{p}) = (i_1, i_2, \ldots, i_m)$ and we will say $\underline{p}$ has *height* $(i_1, i_2, \ldots, i_m)$.

Let $\mathcal{C}_k$ denote the totally ordered chain on $[k]$. If, for each i, the i^{th} component scheme has d_i classes, then the vector-valued height function ℓ is an order-preserving map from the product poset $\mathcal{P}$ onto the product of chains $\mathcal{C} = \mathcal{C}_{d_1} \times \cdots \times \mathcal{C}_{d_m}$. (We will also use $\trianglelefteq$ to denote the partial order on $\mathcal{C}$.) For $\underline{j} = (j_1, \ldots, j_m) \in \mathcal{C}$, $W_{\underline{j}}$ will denote the incidence matrix of objects in $\mathcal{P}$ of height $\underline{j}$ versus objects of maximum height, which are just the elements of X. Similarly, the adjacency matrices and primitive idempotents of the product association scheme are indexed by elements of $\mathcal{C}$. Finally, if $D \subseteq X$ has inner distribution $\mathbf{a}$, the entries of both $\mathbf{a}$ and its MacWilliams transform $\mathbf{a}Q$ are indexed by vectors in $\mathcal{C}$.

Recall that $\mathcal{T} \subseteq \mathcal{C}$ is a *downset* in $(\mathcal{C}, \trianglelefteq)$ if, whenever $\underline{j} \in \mathcal{T}$ and $\underline{i} \trianglelefteq \underline{j}$, we necessarily have $\underline{i} \in \mathcal{T}$. For a fixed downset $\mathcal{T} \subseteq \mathcal{C}$, we say that $D \subseteq X$ is a *poset* $\mathcal{T}$*-design* if there exist constants $\lambda_{\underline{j}}$ $(\underline{j} \in \mathcal{T})$ such that, for every $\underline{p}$ in $\mathcal{P}$, $\ell(\underline{p}) = \underline{j} \in \mathcal{T}$ implies

$$|\{x \in D : \underline{p} \trianglelefteq x\}| = \lambda_{\underline{j}}.$$

In the special case where t is a positive integer and

$$\mathcal{T} = \{\underline{j} \in \mathcal{C} : \ j_1 + \cdots + j_m \leq t\},$$

we will simply say that D is a *poset* t*-design*.

We now present a few examples, limiting attention to products of Hamming lattices and truncated Boolean lattices. Since all Q-posets involved are regular, it suffices to specify $\lambda_{\underline{j}}$ only for the maximal elements $\underline{j} \in \mathcal{T}$. All remaining constants $\lambda_{\underline{j}}$ ($\underline{j} \in \mathcal{T}$) can be computed from these. In such cases, our definition of $\mathcal{T}$-design is highly redundant.

Example 2.2: A *Room square* of side $2v-1$ (see [7]) is an arrangement of all unordered pairs from a set $\mathcal{V}$ of size $2v$ inside a $(2v-1)\times(2v-1)$ array in such a way as to satisfy the following conditions:

- each cell of the array contains at most one unordered pair;
- each unordered pair of elements from $\mathcal{V}$ appears exactly once in the array;
- in each row and in each column, the unordered pairs which occur form a one-factor (i.e., a partition of $\mathcal{V}$ into 2-sets).

Such an object can be viewed as a poset $\mathcal{T}$-design in (the product poset naturally associated to) $H(2,2v-1)\otimes J(2v,2)$ where $\mathcal{T}=\{(0,0),(1,0),(0,1),(1,1),(0,2)\}$ and $\lambda_{(1,1)}=1$. (An extension of this concept is the *Room d-cube* which can be encoded as a $\mathcal{T}$-design in $H(d,2v-1)\otimes J(2v,2)$ with restrictions on its inner distribution.)

More generally, let B denote a (possibly trivial) 2-(v,k,μ) block design. An *α-resolution* of B is a partition of the blocks of B into 1-(v,k,α) designs, called *resolution classes*. Two resolutions are *orthogonal* if any resolution class from one shares at most one block in common with any resolution class from the second. It is not difficult to see that a set of n pairwise orthogonal resolutions of a 2-(v,k,μ) block design is equivalent to a $\mathcal{T}$-design in $H(n,q)\otimes J(v,k)$ where $q=\mu(v-1)/(\alpha(k-1))$ is the number of resolution classes in any resolution,

$$\mathcal{T}=\{(0,0),(1,0),(0,1),(1,1),(0,2)\},$$

$\lambda_{(1,1)}=\alpha$, and the inner distribution $\mathbf{a}$ satisfies $a_{(i,j)}=0$ unless $i=j=0$ or $i\geq d-1$. (This last condition ensures orthogonality.)

Example 2.3: Let B be the set of blocks of a t-(v,k,λ) design. A *resolution of strength* s is a partition of B into s-(v,k,α) designs for some α. Such a structure is equivalent to a $\mathcal{T}$-design (of specified cardinality) in $J(w,1)\otimes J(v,k)$ where

$$w=\frac{\lambda\binom{v}{t}\binom{k}{s}}{\alpha\binom{v}{s}\binom{k}{t}}$$

and

$$\mathcal{T}=\{(0,0),(0,1),\ldots,(0,t),(1,0),(1,1),\ldots,(1,s)\}.$$

The most familiar case is $t=2$, $s=1$. These are called α-resolvable designs. These designs are well-studied (see also [11]). $\square$

As is standard, let $\mathcal{T}^*=\mathcal{T}-\{\underline{0}\}$. We are now prepared to prove the main result of the paper.

Theorem 2.3 *For $1 \leq i \leq m$, let $(Y_i, \mathcal{A}_i)$ be a symmetric d_i-class association scheme with primitive idempotents $E_{i0}, E_{i1} \ldots, E_{id_i}$ (in that order). Let $(\mathcal{P}_i, \preceq_i)$ be a Q-poset for $(Y_i, \mathcal{A}_i)$ with incidence matrices W_{ij} $(0 \leq j \leq d_i)$.*

Let $(X, \mathcal{A})$ be the product association scheme with primitive idempotents $E_{\underline{j}}$ $(\underline{j} \in \mathcal{C})$. For each such $\underline{j}$, define

$$W_{\underline{j}} = \otimes_{i=1}^{m} W_{ij_i}.$$

Let $(\mathcal{P}, \trianglelefteq)$ denote the product poset constructed as above from the $(\mathcal{P}_i, \preceq_i)$. Let $\mathcal{T}$ be any downset in $\mathcal{C}$. Then for $D \subseteq X$ with characteristic vector χ_D, the following are equivalent:

(i) *D is a poset $\mathcal{T}$-design in $(\mathcal{P}, \trianglelefteq)$;*

(ii) *there exist constants $\lambda_{\underline{j}}$, $\underline{j} \in \mathcal{T}$ such that $W_{\underline{j}} \chi_D = \lambda_{\underline{j}} \mathbf{1}$;*

(iii) *for $\underline{j} \in \mathcal{T}^*$, $E_{\underline{j}} \chi_D = 0$ (i.e., D is a Delsarte $\mathcal{T}^*$-design);*

(iv) *for $\underline{j} \in \mathcal{T}^*$, $(\mathbf{a}Q)_{\underline{j}} = 0$.*

Proof: The equivalence of **(i)** and **(ii)** follows from the fact that $W_{\underline{j}}$ is the incidence matrix of objects of height $\underline{j}$ versus objects in X in the product poset $(\mathcal{P}, \trianglelefteq)$. The equivalence of **(iii)** and **(iv)** is again entirely standard.

(ii)$\Rightarrow$(iii): Let $\underline{j} \in \mathcal{T}^*$ and let $\mathbf{e} = \otimes_{i=1}^{m} \mathbf{e}_i$ be any row of $E_{\underline{j}} = \otimes_{i=1}^{m} E_{ij_i}$ where each $\mathbf{e}_i$ is a row of E_{ij_i}. Then $\mathbf{e}$ is orthogonal to the all-ones vector $\mathbf{1}$ since $\underline{j} \neq \underline{0}$. Since each $(\mathcal{P}_i, \preceq_i)$ is a Q-poset for $(Y_i, \mathcal{A}_i)$, there exist vectors $\mathbf{r}_1, \ldots, \mathbf{r}_m$ such that $\mathbf{e}_i = \mathbf{r}_i W_{ij_i}$. So we have

$$0 = \mathbf{e}\mathbf{1}^T = (\otimes_i \mathbf{r}_i W_{ij_i})(\otimes_i \mathbf{1}^T) = (\otimes_i \mathbf{r}_i)(\otimes_i W_{ij_i} \mathbf{1}^T) = \rho_1 \rho_2 \cdots \rho_m (\otimes_i \mathbf{r}_i) \mathbf{1}^T$$

where ρ_i is the (constant) row sum of W_{ij_i}. This shows that $(\otimes_i \mathbf{r}_i)\mathbf{1}^T = 0$. Now we are able to compute the inner product $\langle \mathbf{e}, \chi_D \rangle$:

$$(\otimes_i \mathbf{r}_i W_{ij_i}) \chi_D^T = (\otimes_i \mathbf{r}_i) W_{\underline{j}} \chi_D^T = (\otimes_i \mathbf{r}_i) \lambda_{\underline{j}} \mathbf{1}^T$$

where $\lambda_{\underline{j}}$ is as purported in **(ii)**. Putting this together with the above, we get $\langle \mathbf{e}, \chi_D \rangle = 0$. Since $\mathbf{e}$ was an arbitrary row of $E_{\underline{j}}$, we obtain $E_{\underline{j}} \chi_D = 0$ as desired.

(iii)$\Rightarrow$(ii): Let $\underline{j} \in \mathcal{T}^*$ and assume $E_{\underline{k}} \chi_D = 0$ for all $\underline{k} \in \mathcal{T} - \{\underline{0}\}$ with $\underline{k} \trianglelefteq \underline{j}$. Take any row $\mathbf{w} = \otimes_i \mathbf{w}_i$ of $W_{\underline{j}}$. By axiom **(3)** for the Q-poset $(\mathcal{P}_i, \preceq_i)$, we can write

$$\mathbf{w}_i = \sum_{k=0}^{j_i} \mathbf{e}_{ik}$$

for each $1 \leq i \leq m$ where $\mathbf{e}_{ik} = \mathbf{w}_i E_{ik} \in \mathsf{rowsp}\, E_{ik}$. Note that $\mathbf{e}_{i0} = \alpha_i \mathbf{1}$ where α_i is independent of the choice of row $\mathbf{w}_i$. (This follows from the observation that

$$\mathsf{rowsum}\, W_{ij_i} = \langle \mathbf{w}_i, \mathbf{1} \rangle = \sum_{k=0}^{j_i} \langle \mathbf{e}_{ik}, \mathbf{1} \rangle = \alpha_i |Y_i|.)$$

Hence

$$\langle \mathbf{w}, \chi_D \rangle = \sum_{\underline{k} \,\le\, \underline{j}} \langle \otimes_i \mathbf{e}_{ik_i}, \chi_D \rangle = \langle \otimes_i \mathbf{e}_{i0}, \chi_D \rangle$$

since, for $\underline{k} \neq \underline{0}$, we have $E_{\underline{k}} \chi_D = 0$ and $\otimes_i \mathbf{e}_{ik_i}$ is in the rowspace of $E_{\underline{k}}$. So we have

$$\langle \mathbf{w}, \chi_D \rangle = \alpha_1 \alpha_2 \cdots \alpha_m \langle \mathbf{1}, \chi_D \rangle = \alpha_1 \alpha_2 \cdots \alpha_m |D|,$$

which is independent of the row $\mathbf{w}$ chosen. Hence this final quantity is our $\lambda_{\underline{j}}$ and we have proven $W_{\underline{j}} \chi_D = \lambda_{\underline{j}} \mathbf{1}$. □

An immediate consequence of this result is a linear programming bound for any family of combinatorial objects which can be characterized by condition **(i)** of the theorem. Specifically, one may minimize $|D| = \sum a_{\underline{i}}$ subject to the inequalities of Theorem 1.1 and the equations given by condition **(iv)**, yielding a lower bound on the size of D.

Example 2.4: In [10], Levenshtein introduces the concept of a *split orthogonal array.* Given q, n_1, n_2, t_1, t_2, we wish to find an $M \times (n_1 + n_2)$ array with entries in $\mathcal{Q} = \{0, 1, \ldots, q-1\}$ such that, upon choosing any t_1 columns from among the first n_1 columns and any t_2 columns from among the last n_2 columns, all $(t_1 + t_2)$-tuples over the alphabet $\mathcal{Q}$ occur equally often.

This is equivalent to a $\mathcal{T}$-design in the product scheme $H(n_1, q) \otimes H(n_2, q)$ where

$$\mathcal{T} = \{(i_1, i_2) : 0 \le i_1 \le t_1,\ 0 \le i_2 \le t_2\}.$$

For such objects, Delsarte's linear programming bound (Thm. 1.1) specialises to [10, Thm. 6.3] and our main theorem above appears (in part) as [10, Thm. 5.9]. □

Example 2.5: Mixed-level orthogonal arrays are studied by Sloane and Stufken in [17]. A *mixed-level orthogonal array* $OA(M, q_1^{n_1} q_2^{n_2} \cdots q_m^{n_m}, t)$ *of strength* t is an $M \times n$ matrix ($n = \sum n_i$) whose columns are broken into m blocks — the i^{th} block contains n_i columns and has entries from $\{0, 1, \ldots, q_i - 1\}$ — having the property that, upon choosing any t columns, any two t-tuples which could possibly occur (given the alphabets) appear equally often. These designs are of particular interest in industrial engineering.

Observe that the above definition is equivalent to a poset t-design in a product of Hamming lattices. Hence, by the result above, it is also equivalent to a Delsarte $\mathcal{T}^*$-design in the scheme $H(n_1, q_1) \otimes \cdots \otimes H(n_m, q_m)$ where $\mathcal{T} = \{\underline{j} : \sum j_i \le t\}$. In this case, Delsarte's linear programming bound appears as [17, Theorem 1] and the theorem above specialises to imply [17, Theorem 2]. □

Example 2.6: In the product of Johnson schemes $J(v_1, k_1) \otimes J(v_2, k_2)$, consider Delsarte $\mathcal{T}^*$-designs for $\mathcal{T} = \{(i_1, i_2) :\ i_1 + i_2 \le t\}$. These are called *mixed t-designs* and are studied in [11] where the results established here are applied to that case. In combinatorial terms, a mixed t-$(v_1, k_1, v_2, k_2, \Lambda)$ design is a collection D of ordered pairs (b_1, b_2) where b_i is a k_i-subset of a fixed set $\mathcal{V}_i$ of v_i points having the property that, for any t_1, t_2 with $t_1 + t_2 \le t$, there is a constant Λ_{t_1,t_2} such that the number of pairs (b_1, b_2) in D with $S_i \subseteq b_i$ is independent of the choice of S_1 and S_2 for any subsets $S_i \subseteq \mathcal{V}_i$ satisfying $|S_i| = t_i$. The equivalence is established using the above theorem via the obvious characterisation in terms of poset t-designs in the product of two truncated Boolean lattices. See [11] for a construction of such designs from ordinary t-designs using the (generalised) Assmus-Mattson Theorem [6]. □

Example 2.7: Consider a product scheme of the form $H(n,q) \otimes J(v,k)$. The associated poset is a product of a Hamming lattice and a truncated Boolean lattice. Let us call a poset t-design in this poset a *fused orthogonal array design* of strength t. The elements of D are ordered pairs with first coordinates forming an orthogonal array and second coordinates forming a combinatorial t-design. If we specify any i positions in the first coordinate, the value of the n-tuples in these positions determines a partition of D and this must be a resolution of the design formed by the second coordinates into q^i designs of equal size and strength (at least) $t - i$. Similarly, if we specify any i symbols and isolate those pairs whose second coordinate contains these points, the corresponding first coordinates form an orthogonal array of strength at least $t - i$.

A simple example is the following 2-design in $H(4,2) \otimes J(4,3)$:

$$
\begin{array}{ll}
(0000, \{1,2,3\}), & (1111, \{1,2,3\}), \\
(0011, \{1,2,4\}), & (1100, \{1,2,4\}), \\
(0101, \{1,3,4\}), & (1010, \{1,3,4\}), \\
(1001, \{2,3,4\}), & (0110, \{2,3,4\}).
\end{array}
$$
□

3 Bounds for designs in products of Q-polynomial schemes

3.1 The Delsarte bound

One well-known theorem [4, Theorem 5.19] of Delsarte gives a lower bound on the size of a t-design in a Q-polynomial association scheme. This bound has both the Rao bound for orthogonal arrays and the bound of Ray-Chaudhuri and Wilson for block designs as special cases. In this section, we generalise this bound further to the case of $\mathcal{T}$-designs in products of Q-polynomial association schemes.

Let $(X, \mathcal{A})$ be an association scheme with $v = |X|$. Consider the *Krein parameters* $q_{ij}(k)$ defined by

$$E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{d} q_{ij}(k) E_k.$$

The well-known Krein condition states that each $q_{ij}(k) \geq 0$. We say $(X, \mathcal{A})$ is Q*-polynomial* if the following conditions are satisfied for all i, j, k:

$$(k > i + j \quad \Rightarrow \quad q_{ij}(k) = 0) \qquad \text{and} \qquad (i + j \leq d \quad \Rightarrow \quad q_{ij}(i + j) > 0).$$

(See [1, Sec. III.1] or [2, Sec. 2.7] for much more information.)

Let $\mathcal{C}$ be the poset formed by the product of chains $\mathcal{C}_{d_1} \times \cdots \times \mathcal{C}_{d_m}$. For $\mathcal{E} \subseteq \mathcal{C}$, define

$$\mathcal{E} + \mathcal{E} = \{(i_1 + j_1, \ldots, i_m + j_m) : \ \underline{i}, \underline{j} \in \mathcal{E}\}.$$

For $\underline{j} \in \mathcal{C}$, let $f_{\underline{j}} = \ \mathsf{rank}\, E_{\underline{j}}$.

Lemma 3.1 *Let $(X, \mathcal{A})$ be the product of association schemes $(Y_h, \mathcal{A}_h)$ $(1 \leq h \leq m)$. Let the Krein parameters of $(Y_h, \mathcal{A}_h)$ be denoted by $q_{i_h j_h}(k_h)$. Then for $\underline{i}, \underline{j}, \underline{k} \in \mathcal{C}$, the corresponding Krein parameter of $(X, \mathcal{A})$ is given by*

$$q_{\underline{i}\underline{j}}(\underline{k}) = \prod_{h=1}^{m} q_{i_h j_h}(k_h).$$

Furthermore, if each component scheme is Q-polynomial, then we have $q_{\underline{i}\underline{j}}(\underline{k}) = 0$ whenever there exists an h for which $k_h > i_h + j_h$. $\square$

Theorem 3.2 (Delsarte Bound) *Let $(X, \mathcal{A})$ be the product of Q-polynomial association schemes $(Y_i, \mathcal{A}_i)$ $(1 \leq i \leq m)$. Let $\mathcal{T}$ be a downset in $\mathcal{C}$ and let $D \subseteq X$ be a Delsarte $\mathcal{T}^*$-design. If $\mathcal{E} \subseteq \mathcal{C}$ satisfies $(\mathcal{E} + \mathcal{E}) \cap \mathcal{C} \subseteq \mathcal{T}$, then*

$$|D| \geq \sum_{\underline{j} \in \mathcal{E}} f_{\underline{j}}.$$

Moreover, if equality holds, then, for $\underline{\ell} \neq \underline{0}$,

$$\sum_{\underline{j} \in \mathcal{E}} Q_{\underline{\ell}\underline{j}} = 0$$

whenever D contains a pair of $\underline{\ell}$-related elements.

Proof: We begin with a derivation of the dual of Delsarte's linear program for $\mathcal{T}^*$-designs. Each matrix $M \in \mathbb{A}$ may be expanded in the form

$$M = v \sum_{\underline{j} \in \mathcal{C}} \beta_{\underline{j}} E_{\underline{j}}.$$

If we change bases, we have

$$M = \sum_{\underline{i} \in \mathcal{C}} \alpha_{\underline{i}} A_{\underline{i}}$$

where

$$\alpha_{\underline{i}} = \sum_{\underline{j} \in \mathcal{C}} Q_{\underline{i}\underline{j}} \beta_{\underline{j}} \qquad (\underline{i} \in \mathcal{C}).$$

Consider only matrices M satisfying the conditions

(a) M is a non-negative matrix;

(b) $\beta_{\underline{j}} \leq 0$ for $\underline{j} \notin \mathcal{T}$;

(c) $\beta_{\underline{0}} = 1$.

Let D be any Delsarte $\mathcal{T}^*$-design in $(X, \mathcal{A})$. Since M is non-negative, we have

$$\chi_D^T M \chi_D = \sum_{\underline{i} \in \mathcal{C}} \alpha_{\underline{i}} \, \chi_D^T A_{\underline{i}} \chi_D \geq \alpha_{\underline{0}} \, \chi_D^T A_{\underline{0}} \chi_D = \alpha_{\underline{0}} \, |D|.$$

On the other hand, we have

$$\begin{aligned} \chi_D^T M \chi_D &= v \sum_{\underline{j} \in \mathcal{C}} \beta_{\underline{j}} \, \chi_D^T E_{\underline{j}} \chi_D \\ &= v \beta_{\underline{0}} \, \chi_D^T E_{\underline{0}} \chi_D + v \sum_{\underline{j} \in \mathcal{T}^*} \beta_{\underline{j}} \, \chi_D^T E_{\underline{j}} \chi_D + v \sum_{\underline{j} \notin \mathcal{T}} \beta_{\underline{j}} \, \chi_D^T E_{\underline{j}} \chi_D. \end{aligned}$$

Since $\beta_{\underline{0}} = 1$, $E_{\underline{0}} = \frac{1}{v} J$ and $E_{\underline{j}} \chi_D = 0$ for $\underline{j} \in \mathcal{T}^*$, this reduces to

$$\chi_D^T M \chi_D = |D|^2 + v \sum_{\underline{j} \notin \mathcal{T}} \beta_{\underline{j}} \, \chi_D^T E_{\underline{j}} \chi_D,$$

which is bounded above by $|D|^2$ since, for $\underline{j} \notin \mathcal{T}$, $\beta_{\underline{j}} \leq 0$ and $\chi_D^T E_{\underline{j}} \chi_D \geq 0$. Putting the two inequalities together gives the bound

$$\alpha_{\underline{0}} \leq |D|.$$

In this way, each matrix $M \in \mathbb{A}$ satisfying conditions **(a)**-**(c)** gives a lower bound on the size of a $\mathcal{T}$-design. (Notice that $\alpha_{\underline{0}}$ is the value on the diagonal of M.)

Now let $\mathcal{E} \subseteq \mathcal{T}$ be as hypothesized. Consider the matrix

$$N = \sum_{\underline{j} \in \mathcal{E}} E_{\underline{j}}.$$

If we square each entry of this matrix, we obtain the non-negative matrix

$$N \circ N = \sum_{\underline{h} \in \mathcal{E}} \sum_{\underline{i} \in \mathcal{E}} E_{\underline{h}} \circ E_{\underline{i}} = \sum_{\underline{j} \in \mathcal{C}} \left(\frac{1}{v} \sum_{\underline{h} \in \mathcal{E}} \sum_{\underline{i} \in \mathcal{E}} q_{\underline{h}\underline{i}}(\underline{j}) \right) E_{\underline{j}}.$$

Notice that, by Lemma 3.1, $q_{\underline{h}\underline{i}}(\underline{j}) = 0$ whenever $\underline{h}, \underline{i} \in \mathcal{E}$ and $\underline{j} \notin \mathcal{T}$. Thus the coefficient of $E_{\underline{j}}$ vanishes whenever $\underline{j} \notin \mathcal{T}$. On the other hand, since $q_{\underline{h}\underline{i}}(\underline{0}) = \delta_{\underline{h},\underline{i}} f_{\underline{h}}$, we have $\frac{1}{v}\sum_{\mathcal{E}} f_{\underline{h}}$ as the coefficient of $E_{\underline{0}}$ in this expansion.

Let

$$\gamma = \frac{v^2}{\sum_{\underline{h}\in\mathcal{E}} f_{\underline{h}}},$$

and take $M = \gamma(N \circ N)$. Then M is non-negative and, if we expand

$$M = v \sum_{\underline{j}\in\mathcal{C}} \beta_{\underline{j}} E_{\underline{j}},$$

we obtain $\beta_{\underline{0}} = 1$ and $\beta_{\underline{j}} = 0$ for $\underline{j} \notin \mathcal{T}$. So M satisfies conditions **(a)**, **(b)**, and **(c)**. Consequently, for any $\mathcal{T}$-design D, the size of D is bounded below by the common value of the diagonal entries of M. The diagonal entries of $E_{\underline{h}}$ are all equal to $f_{\underline{h}}/v$; the diagonal entries of N are therefore $\frac{1}{v}\sum_{\mathcal{E}} f_{\underline{h}}$. So each diagonal entry of M is equal to

$$\gamma \left(\frac{\sum_{\underline{h}\in\mathcal{E}} f_{\underline{h}}}{v} \right)^2 = \sum_{\underline{h}\in\mathcal{E}} f_{\underline{h}}.$$

This proves the inequality

$$|D| \geq \sum_{\underline{h}\in\mathcal{E}} f_{\underline{h}}.$$

Now if equality holds in this last inequality, then we must have equality throughout. In particular, for each $\underline{\ell} \neq \underline{0}$, we must have $\alpha_{\underline{\ell}}(\chi_D^T A_{\underline{\ell}} \chi_D) = 0$. In other words, if $\underline{\ell} \neq \underline{0}$, and D contains two $\underline{\ell}$-related elements, then

$$\sum_{\underline{j}\in\mathcal{C}} \beta_{\underline{j}} Q_{\underline{\ell}\underline{j}} = 0. \ \square$$

Example 3.1: For split orthogonal arrays, we obtain the bound

$$M \geq \sum_{i=0}^{\lfloor t_1/2 \rfloor} \sum_{j=0}^{\lfloor t_2/2 \rfloor} \binom{n_1}{i}\binom{n_2}{j}(q-1)^{i+j}$$

on the number of rows, M, in the array. Obviously, the right-hand side factors into the product of the ordinary Rao bounds for $OA(t_i, n_i, q)$, $i = 1, 2$. This result is implied by Theorem 6.3 in [10]. $\square$

Example 3.2: For mixed-level orthogonal arrays, the theorem above gives

$$\sum_{i_1+\cdots+i_m \leq \lfloor t/2 \rfloor} \prod_{h=1}^{m} \binom{n_h}{i_h}(q_h - 1)^{i_h}$$

as a lower bound on the number of rows of the array. This bound was already derived by elementary means in [9].

Now if equality holds in the bound, the second statement of the theorem forces many entries of the inner distribution to equal zero. For example, in the case $t = 2$, if $\underline{\ell} \neq \underline{0}$ and D contains a pair of $\underline{\ell}$-related elements, then $\ell_1 q_1 + \cdots \ell_m q_m = |D|$. For $t = 2$ and 4, these restrictions were already found by Mukerjee and Wu [15]. □

Example 3.3: For mixed t-designs in $J(v_1, k_1) \otimes J(v_2, k_2)$, we may take $\mathcal{E} = \{(i_1, i_2) : i_1 + i_2 \leq u\}$ where $u = \lfloor t/2 \rfloor$. This gives the bound

$$|D| \geq \sum_{0 \leq i_1 + i_2 \leq u} \left[\binom{v_1}{i_1} - \binom{v_1}{i_1 - 1} \right] \left[\binom{v_2}{i_2} - \binom{v_2}{i_2 - 1} \right],$$

which simplifies to

$$|D| \geq \binom{v_1 + v_2}{u} - \binom{v_1 + v_2}{u - 1}.$$

For example, for $t = 2$, we find $|D| \geq v_1 + v_2 - 1$ and for $t = 4$, $|D| \geq \frac{1}{2}(v_1 + v_2 - 3)(v_1 + v_2)$. □

Example 3.4: For a fused orthogonal array design of strength t in $H(n, q) \otimes J(v, k)$, we obtain

$$|D| \geq \sum_{j_1 + j_2 \leq u} \binom{n}{j_1} (q-1)^{j_1} \left[\binom{v}{j_2} - \binom{v}{j_2 - 1} \right]$$

where $u := \lfloor t/2 \rfloor$. This can be written in terms of a hypergeometric series:

$$|D| \geq \binom{v}{u} \, {}_2F_1 \left[\begin{matrix} -u, -n \\ v - u + 1 \end{matrix} ; \; q - 1 \right]. \; \square$$

3.2 The degree bound and association schemes induced by designs

Continuing the treatment in parallel to that of designs in Q-polynomial association schemes, we prove in this section a relationship between the size of $\mathcal{T}$ and the degree s of a $\mathcal{T}$-design, focusing on the case where this bound is tight.

Let D be a design in an association scheme $(X, \mathcal{A})$. Let $\mathbb{A}$ denote the Bose-Mesner algebra of $(X, \mathcal{A})$. For $M \in \mathbb{A}$, denote by $\bar{M}$ the submatrix of M whose rows and columns are indexed by the elements of D. In the case where $(X, \mathcal{A})$ is Q-polynomial and D has degree s and strength t, a well-known result [4, Theorem 5.25] of Delsarte states that $\lfloor t/2 \rfloor \leq s$ and, if $t \geq 2s - 2$, then the vector space $\bar{\mathbb{A}}$ of submatrices $\{\bar{M} : M \in \mathbb{A}\}$ is the Bose-Mesner algebra of a (Q-polynomial) association scheme. We express this by saying that D *induces an association scheme* in $(X, \mathcal{A})$.

Now let D be a Delsarte $\mathcal{T}^*$-design in a product $(X, \mathcal{A})$ of Q-polynomial association schemes. Recall that the *degree* of D is the number of indices $\underline{i} \neq \underline{0}$ for which $\bar{A}_{\underline{i}}$ is a non-zero matrix. Call D a *tight design* in $(X, \mathcal{A})$ if, for some $\mathcal{E}$, equality holds in the bound of Theorem 3.2.

Theorem 3.3 *Let D be a Delsarte $\mathcal{T}^*$-design of degree s in the product $(X, \mathcal{A})$ of Q-polynomial association schemes. If $(\mathcal{E} + \mathcal{E}) \cap \mathcal{C} \subseteq \mathcal{T}$, then*

$$|\mathcal{E}| \leq s + 1.$$

Moreover, if equality holds, then D is a tight design and D induces an association scheme in $(X, \mathcal{A})$.

Secondly, if $|\mathcal{E}| = s$, then either D is a tight design or D induces an association scheme in $(X, \mathcal{A})$.

Proof: The proof is modelled after the proof of Theorem 5.25 in [4].

Let S be a $v \times v$ matrix diagonalising the Bose-Mesner algebra $\mathbb{A}$ of the product scheme. Then S may be partitioned into $v \times f_{\underline{j}}$ submatrices $S_{\underline{j}}$ where the columns of $S_{\underline{j}}$ form an orthonormal basis for $\mathsf{colsp}\, E_{\underline{j}}$ $(\underline{j} \in \mathcal{C})$. Clearly $S_{\underline{j}} S_{\underline{j}}^T = E_{\underline{j}}$. Let $H_{\underline{j}}$ be the submatrix of $S_{\underline{j}}$ obtained by restriction to the rows indexed by elements of D. Then $H_{\underline{j}} H_{\underline{j}}^T = \bar{E}_{\underline{j}}$.

For $\underline{i}, \underline{j} \in \mathcal{C}$, we have (Delsarte [4, Theorem 3.15]):

$$q_{\underline{ij}}(\underline{k})(\mathbf{a}Q)_{\underline{k}} = 0 \quad \text{for all} \quad \underline{k} \in \mathcal{C} - \{\underline{0}\}$$

if and only if

$$H_{\underline{i}}^T H_{\underline{j}} = \begin{cases} 0, & \text{if } \underline{i} \neq \underline{j}; \\ |D| I, & \text{if } \underline{i} = \underline{j}. \end{cases}$$

Consider the vector space $\bar{\mathbb{A}} = \{\bar{M} : M \in \mathbb{A}\}$. This is spanned by the matrices $\{\bar{A}_{\underline{i}} : \underline{i} \in \mathcal{C}\}$ and so has dimension $s + 1$ where s is the degree of D.

For $\underline{i}, \underline{j} \in \mathcal{E}$, we apply Lemma 3.1 to conclude that $q_{\underline{ij}}(\underline{k}) = 0$ whenever $\underline{k} \notin \mathcal{T}$. As $\bar{E}_{\underline{i}} \bar{E}_{\underline{j}} = H_{\underline{i}} H_{\underline{i}}^T H_{\underline{j}} H_{\underline{j}}^T$, we obtain

$$\bar{E}_{\underline{i}} \bar{E}_{\underline{i}} = |D| \, \bar{E}_{\underline{i}} \tag{6}$$

and (assuming $\underline{i} \neq \underline{j}$)

$$\bar{E}_{\underline{i}} \bar{E}_{\underline{j}} = 0. \tag{7}$$

So these $|\mathcal{E}|$ members of $\bar{\mathbb{A}}$ are linearly independent. This proves that $|\mathcal{E}| \leq s + 1$. If equality holds, then (by (6) and (7)) $\bar{\mathbb{A}}$ has a basis of mutually orthogonal idempotents and hence is closed under matrix multiplication. It is easy to see that this vector space is closed under Schur multiplication and contains I and J. Therefore it is the Bose-Mesner algebra of an association scheme (see [2, Theorem 2.6.1]).

Write $\mathcal{E} = \{\underline{i}_1, \ldots, \underline{i}_r\}$. Define

$$G_{\mathcal{E}} = [H_{\underline{i}_1}\ H_{\underline{i}_2}\ \cdots\ H_{\underline{i}_r}].$$

$G_{\mathcal{E}}$ has $|D|$ rows and $\sum_{j=1}^{r} f_{\underline{i}_j}$ columns. From the preceding observations, we see that $G_{\mathcal{E}}^T G_{\mathcal{E}} = |D| I$. (This gives an alternative proof of the bound in Theorem 3.2.)

If D is not a tight design, we may add columns to form an orthogonal matrix $(1/\sqrt{|D|})G$ with

$$G = [G_{\mathcal{E}}\ K].$$

Then $GG^T = |D|I$, yielding

$$KK^T = |D|I - \sum_{j=1}^{r} H_{\underline{i}_j} H_{\underline{i}_j}^T,$$

Now $H_{\underline{i}} H_{\underline{i}}^T = \bar{E}_{\underline{i}}$. Thus $KK^T \in \bar{\mathbb{A}}$ and, by Equations (6) and (7),

$$(\frac{1}{|D|} KK^T)\bar{E}_{\underline{i}} = 0$$

for $\underline{i} \in \mathcal{E}$. Since $K^T K = I$, KK^T is not the zero matrix. So we have $|\mathcal{E}| + 1$ linearly independent matrices in $\bar{\mathbb{A}}$. Thus, if $|\mathcal{E}| = s + 1$, D must be a tight design.

Finally, assume $|\mathcal{E}| = s$ and that D is not a tight design. If we take $c = 1/|D|$, the matrices

$$\{c\bar{E}_{\underline{j}} : \underline{j} \in \mathcal{E}\} \cup \{cKK^T\}$$

form a basis of primitive idempotents for $\bar{\mathbb{A}}$. So $\bar{\mathbb{A}}$ is closed under both ordinary and Schur multiplication and contains I and J. Consequently, $\bar{\mathbb{A}}$ is the Bose-Mesner algebra of an association scheme. □

For example, if D is a t-design in a product of m Q-polynomial schemes, then the degree of D is at least the size of

$$\mathcal{E} = \{(i_1, \ldots, i_m) : 1 \le i_1 + \cdots i_m \le t/2,\ \text{all } i_j \ge 0\}.$$

This gives the bound $s + 1 \ge \binom{m+u}{m}$ where $u = \lfloor t/2 \rfloor$. For example, if D is a mixed-level orthogonal array of strength two in $H(n_1, q_1) \otimes \cdots \otimes H(n_m, q_m)$ then $a_{\underline{i}}$ is non-zero for at least m non-zero index vectors $\underline{i}$.

Remarks:

(i) If D is a tight design, the bound $|\mathcal{E}| \le s$ may fail. For example, consider a projective plane 2-$(n^2+n+1, n+1, 1)$. Delete a block x and let D denote the set of n^2+n remaining blocks, viewed as a 2-design in $J(n+1, 1) \otimes J(n^2, n)$. Then D has degree two in this product scheme yet the set $\mathcal{E} = \{(0,0), (0,1), (1,0)\}$ satisfies $(\mathcal{E} + \mathcal{E}) \cap \mathcal{C} \subseteq \mathcal{T}$. See [11] for an investigation of tight 2-designs in $J(v_1, k_1) \otimes J(v_2, k_2)$.

(ii) In contrast to the situation where $(X, \mathcal{A})$ is Q-polynomial, the induced association scheme found above is not necessarily Q-polynomial, even though we have assumed that each component scheme is Q-polynomial.

4 Summary

Two decades ago, a theory was established for the study of t-designs in Q-polynomial association schemes. In certain cases, these objects have useful combinatorial interpretations. Motivated by several applications, the present paper begins to establish a theory of $\mathcal{T}$-designs in products of Q-polynomial association schemes. In the language of Q-posets, a combinatorial characterisation (Theorem 2.3) of such objects is given. Independent of this characterisation, two results are proved for designs in products of Q-polynomial association schemes which are analogues of Delsarte's results on t-designs. In addition, the linear programming bound for such designs comes "for free".

A variety of questions arise from this investigation. Which association schemes admit a Q-poset? Must such a scheme be Q-polynomial? Is the bound in Theorem 3.2 the best possible? What are the tight designs? Do any new association schemes arise from Theorem 3.3?

The theory in the latter part of the paper can be extended to arbitrary association schemes, but will only be non-trivial in cases where a large number of Krein parameters vanish. See [13] for an application of such techniques to a problem in numerical integration. It also seems useful to focus on particular classes of product schemes, such as products of Hamming schemes (split orthogonal arrays and mixed orthogonal arrays), products of Johnson schemes (mixed block designs [11] and split designs [12]), or $H(n,q) \otimes J(v,k)$ (Room d-cubes and their generalisations).

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