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Characterizing completely regular codes from an algebraic viewpoint

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Abstract. The class of completely regular codes includes not only some of the most important error-correcting codes, such as perfect codes and uniformly packed codes, but also a number of substructures fundamental to the study of distance-regular graphs themselves.

In a companion paper, we study products of completely regular codes and codes whose parameters form arithmetic progressions. This family of completely regular codes, while quite special in one sense, contains some very important examples and exhibits some of the nicest features of the larger class. Here, we approach these features from an algebraic viewpoint, exploring Q-polynomial properties of completely regular codes and introducing Leonard completely regular codes.

After reformulating some basic background on completely regular codes in a unified way, we propose the study of a certain class of codes where the eigenspaces of the code are naturally arranged in a linear order. In addition to the arithmetic codes of the companion paper, this highly structured class of codes, which we call Leonard completely regular codes, includes other interesting examples and we propose their classification in the Hamming graphs. The main result of the paper shows that the Leonard condition is equivalent to the presence of a certain Leonard pair acting on the outer distribution module. This connection has impact in two directions. First, the Leonard pairs have been classified by Terwilliger and we gain quite a bit of information about the algebraic structure of any code in our class. But also this gives a new setting for the study of Leonard pairs, one closely related to the classical one – where a Leonard pair arises from each thin/dual-thin irreducible module of a Terwilliger algebra of some P- and Q-polynomial association scheme — yet not previously studied. It is particularly interesting that the Leonard pair associated to some code C may belong to one family in the Askey scheme while the distance-regular graph containing the code may belong to another.

1. Introduction

The study of digital error-correcting codes includes as an important and intriguing sub-topic the analysis and classification of highly regular codes. These include the perfect codes as well as several phenomenal families such as the Kerdock codes,

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the Delsarte-Goethals codes, and the Reed-Muller codes. One motivation for this branch of coding theory has always been a well-studied but mysterious connection to finite groups. Optimal codes tend to have a great deal of symmetry (as is often true in optimization problems which themselves are defined in a symmetric way), and several finite simple groups – namely the Mathieu groups – play an important role in the classification of perfect codes.

The class of completely regular codes, which properly contains both the perfect codes and the uniformly packed codes, has not received a great deal of attention in recent years. This class also contains the extended Preparata and extended Kasami codes, as well as the Kasami codes [**2**, p. 356]. Our view is that these codes deserve further study, not only because of their connection to highly symmetric codes and codes with large minimum distance, but also because of a key role that completely regular codes play in the study of distance-regular graphs. A theorem of Brouwer, et al. [**2**, p. 353] states that every distance-regular graph on a prime power number of vertices admitting an elementary abelian group of automorphisms which acts transitively on its vertices is a coset graph of some additive completely regular code in some Hamming graph (with some conference graphs as exceptions). This gives another reason why a careful study of completely regular codes in Hamming graphs (and, more generally, in distance-regular graphs) is central to the study of association schemes.

In a companion paper [**10**], we study products of completely regular codes and codes whose parameters form arithmetic progressions. This family of completely regular codes, while quite special in one sense, contains some very important examples and exhibits some of the nicest features of the larger class. Here, we approach these features from an algebraic viewpoint, leading to the definition of *Leonard completely regular codes*, which enjoy a certain "Q-polynomial property".

We first summarize the basic structure of the outer distribution module of a completely regular code. Then, employing a simple lemma concerning eigenvectors in association schemes, we propose to study the tightest case, where the eigenvectors of the code can be ordered so that the j^{th} one is expressible as an entrywise polynomial of degree exactly j in the eigenvector with subscript one in the ordering (see Definition 4.1 for a precise formulation). In addition to the arithmetic codes of the companion paper, this highly structured class includes other beautiful examples and we propose the classification of Leonard completely regular codes in the Hamming graphs. A key result is Theorem 5.4 which finds that the Leonard condition is equivalent to the presence of a certain Leonard pair. This connection has impact in two directions. First, the Leonard pairs are classified and we gain quite a bit of information about the algebraic structure of any code in our class. But also this gives a new setting for the study of Leonard pairs, one closely related to the classical one where a Leonard pair arises from each thin/dual-thin irreducible module of a Terwilliger algebra of some P- and Q-polynomial association scheme, yet not previously studied. It is particularly interesting that the Leonard pair associated to some code C may belong to one family in the Askey scheme while the distance-regular graph in which the code is found may belong to another.

1.1. Prior work. The results here are the outgrowth of a number of related projects over the years. Not only do we employ here tools from the literature that provide results on distance-regular graphs, codes, association schemes, and Leonard pairs, but we are also aware that there are alternative approaches to some of the ideas presented here. We now briefly review the context in which the present work is being done.

The basic algebraic approach to codes that we take was set out in Delsarte's thesis [**6**]. (Biggs developed some of the theory independently around the same time — see [**2**, p. 349].) We already see the outer distribution matrix of a code in [**6**, Sec. 3.1] and, implicitly, the two bases for the outer distribution module that we will examine in Section 3. These bases also appear implicitly in [**13**] and explicitly in [**11**]. The action of the Bose-Mesner algebra on the outer distribution module is first discussed in [**11**, Sec. 2.1.5] and the outer distribution module itself first appears in [**8**, p. 188]. Codes with Q-polynomial properties begin to appear in [**3**, Sec. 5]. Later, Suzuki [**16**] studied the "Terwilliger algebra" of a code and re-developed some of the material in these earlier papers using the same notation and terminology. Our present investigation has little overlap with [**16**], except that some basic results we present in Section 3 — most of which are implicit in the earlier work we have just outlined — also appear in Section 7 of that paper. We do not consider Terwilliger algebras here.

In this paper, we introduce Leonard pairs into the study of completely regular codes. There is a rapidly growing literature on the subject of Leonard pairs, as more and more applications and connections are discovered. A good starting reference is the recent survey paper of Terwilliger [**20**]. We do not use much of the theory of Leonard pairs in this paper; everything we need can be found in [**17, 18, 19**].

2. Preliminaries and definitions

Here, we briefly recall basic facts and basic terminology and notation from the theory of distance-regular graphs.

2.1. Distance-regular graphs. Suppose that Γ is a finite, undirected, connected graph with vertex set $V\Gamma$. For vertices x and y in $V\Gamma$, let $d(x, y)$ denote the distance between x and y, i.e., the length of a shortest path connecting x and y in Γ. Let D denote the diameter of Γ; i.e., the maximal distance between any two vertices in $V\Gamma$. For $0 \leq i \leq D$ and $x \in V\Gamma$, let $\Gamma_i(x) := \{y \in V\Gamma \mid d(x, y) = i\}$ and put $\Gamma_{-1}(x) := \emptyset$, $\Gamma_{D+1}(x) := \emptyset$. The graph Γ is called *distance-regular* whenever it is a regular graph and there are integers b_i, c_i $(0 \leq i \leq D)$ so that for any two vertices x and y in $V\Gamma$ at distance i, there are precisely c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$. If we let k denote the valency of Γ, it follows that there are exactly $a_i = k - b_i - c_i$ neighbors of y in $\Gamma_i(x)$. The numbers c_i, a_i and b_i are called the *intersection numbers* of Γ and we observe that $c_0 = 0$, $b_D = 0$, $a_0 = 0$, $c_1 = 1$ and $b_0 = k$. The array $\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the *intersection array* of Γ. Set the tridiagonal matrix

$$L(\Gamma) := \begin{pmatrix} a_0 & b_0 & & & \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & b_2 & \\ & & \ddots & \ddots & \ddots \\ & & & c_D & a_D \end{pmatrix}. \tag{2.1}$$

From now on, assume Γ is a distance-regular graph of valency $k \geq 2$ and diameter $D \geq 2$. Define A_i to be the square matrix of size $|V\Gamma|$ whose rows and columns are indexed by $V\Gamma$ with entries

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } d(x,y) = i; \\ 0, & \text{otherwise;} \end{cases} \qquad (0 \le i \le D,\ x, y \in V\Gamma).$$

We refer to A_i as the i^{th} *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. Since Γ is distance-regular we have, for $2 \le i \le D$,

$$AA_{i-1} = b_{i-2}A_{i-2} + a_{i-1}A_{i-1} + c_iA_i$$

so that $A_i = p_i(A)$ for some polynomial $p_i(t)$ of degree i. If we define also $p_{D+1}(t) = (t - a_D)p_D(t) - b_{D-1}p_{D-1}(t)$, then $p_{D+1}(A) = 0$ since $AA_D = a_DA_D + b_{D-1}A_{D-1}$.

By an *eigenvalue* of Γ, we mean an eigenvalue of $A = A_1$. Since Γ has diameter D, it has at least $D+1$ eigenvalues; but since $p_{D+1}(A) = 0$, it has exactly $D+1$ eigenvalues. We denote these eigenvalues by $\theta_0, \ldots, \theta_D$ and maintain the convention that $\theta_0 = k$, the valency of Γ. The matrices A_i act on the space $V := \mathbb{C}^{V\Gamma}$, which is called the *standard module* of Γ. For each $0 \le j \le D$, this action preserves the j^{th} eigenspace of Γ,

$$V_j := \{\mathbf{v} \in V \mid A\mathbf{v} = \theta_j\mathbf{v}\}.$$

(We also call this the "eigenspace belonging to θ_j".) Let E_j $(0 \le j \le D)$ denote the matrix representing orthogonal projection of V onto V_j. Then we have $AE_j = \theta_jE_j$ for $0 \le j \le D$ and each E_j is expressible as a polynomial in A. (When $\theta = \theta_j$ for some j, we will sometimes write $E(\theta)$ in place of E_j when it is convenient to omit the subscript.)

Let $\mathbb{A}$ be the *Bose-Mesner algebra*, the matrix algebra generated by A over the real numbers $\mathbb{R}$. From above, we see that $\{A_i \mid 0 \le i \le D\}$ is a basis for $\mathbb{A}$. As $\mathbb{A}$ is semi-simple and commutative, $\mathbb{A}$ has also a basis of pairwise orthogonal idempotents: one easily verifies that this basis is $\left\{E_0 = \frac{1}{|V\Gamma|}J, E_1, \ldots, E_D\right\}$. We call the E_j the *primitive idempotents* of Γ. As $\mathbb{A}$ is closed under the entrywise (or Hadamard) product $\circ$, there exist real numbers q^ℓ_{ij} $(0 \le i, j, \ell \le D)$, called the *Krein parameters*, such that

$$E_i \circ E_j = \frac{1}{|V\Gamma|}\sum_{\ell=0}^{D} q^\ell_{ij}E_\ell, \quad (0 \le i, j \le D). \tag{2.2}$$

We say that Γ is *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever the following hold for $0 \le i, j, \ell \le D$:

- $q^\ell_{ij} = 0$ unless $|j - i| \le \ell \le i + j$;
- $q^\ell_{ij} \ne 0$ if $\ell = |j - i|$ or $\ell = i + j$.

When these conditions hold for this ordering of the E_j, we call $E_0, E_1, \ldots, E_D$ a *Q-polynomial ordering* of the idempotents. (Equivalently, we may say that the corresponding ordering $\theta_0, \ldots, \theta_D$ of the eigenvalues of Γ is a "Q-polynomial ordering" of the eigenvalues.) But for now, aside from the convention that $\theta_0 = k$, we make no further assumptions at this point about the eigenvalues except that they are distinct.

For each eigenvalue θ of Γ and for each $x \in V\Gamma$, there is a unique normalized eigenvector in the eigenspace of Γ belonging to θ which is constant over each vertex subset $\Gamma_i(x)$; since $E(\theta)$ is expressible as a linear combination of the A_i, this vector is a scalar multiple of column x of $E(\theta)$. It is well-known (and quite useful), however, that the entries of this vector are entirely determined by the intersection array $\iota(\Gamma) := \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ of the graph Γ. Consider the vector

$$u(\theta) := [u_0(\theta) = 1, u_1(\theta), \ldots, u_D(\theta)]^\top$$

of length $D+1$ defined by the following initial conditions and recurrence relation:

$$\begin{gathered} u_0(\theta) = 1,\ u_1(\theta) = \theta/k, \\ c_i u_{i-1}(\theta) + a_i u_i(\theta) + b_i u_{i+1}(\theta) = \theta u_i(\theta) \quad (0 \le i \le D), \\ \text{where } \ u_{-1}(\theta) = u_{D+1}(\theta) = 0. \end{gathered} \tag{2.3}$$

This is a right eigenvector of the tridiagonal matrix $L(\Gamma)$ defined in (2.1) above; the corresponding eigenvalue is θ. We see this by using the equation $AE(\theta) = \theta E(\theta)$ to verify that the $D+1$ distinct entries in column x of $E(\theta)$ must satsify the same three-term recurrence. This shows that the $D+1$ eigenvalues of Γ are precisely the eigenvalues of $L(\Gamma)$. (See, e.g., [**7**, Lemma 11.4.1]). The eigenvector $u(\theta)$ is called the *standard right eigenvector* of Γ belonging to θ. For any vertex x and any eigenvalue $\theta = \theta_j$, $(0 \le j \le D)$, we then obtain column x of $E_j = E(\theta)$: for $y \in \Gamma_i(x)$, its entry in position y is simply

$$\frac{m_j}{|V\Gamma|} u_i(\theta_j)$$

where $m_j := \operatorname{rank} E_j$. From this and (2.2), it follows that

$$m_i m_j\, u_h(\theta_i) u_h(\theta_j) = \sum_{\ell=0}^{D} q_{ij}^{\ell} m_\ell u_h(\theta_\ell) \tag{2.4}$$

for $0 \le h, i, j \le D$.

Remark 2.1. Using (2.4), one may easily detect whether or not Γ is Q-polynomial with respect to any given ordering just by looking at its standard right eigenvectors.

The following fundamental result will be very useful in this paper; it is originally due to Cameron, Goethals, and Seidel [**5**].

Theorem 2.2 ([**5**, Theorem 5.1]). *If* $\mathbf{u} \in V_i$ *and* $\mathbf{v} \in V_j$ *and* $q_{ij}^{\ell} = 0$, *then* $\mathbf{u} \circ \mathbf{v}$ *is orthogonal to* V_ℓ *where* $\mathbf{u} \circ \mathbf{v}$ *denotes the entrywise product of vectors* $\mathbf{u}$ *and* $\mathbf{v}$. □

An elementary proof of this fact can be found in [**12**].

3. The outer distribution module of a completely regular code

Let Γ be a distance-regular graph with distinct eigenvalues $\theta_0 = k, \theta_1, \ldots, \theta_D$. By a *code* in Γ, we simply mean any nonempty subset C of $V\Gamma$. We call C *trivial* if $|C| \le 1$ or $C = V\Gamma$ and *nontrivial* otherwise. For $|C| > 1$, the *minimum distance* of C, $\delta(C)$, is defined as

$$\delta(C) := \min\{\ d(x,y) \mid x, y \in C, x \neq y\ \}$$

and for any $x \in V\Gamma$ the distance $d(x, C)$ from x to C is defined as

$$d(x, C) := \min\{\ d(x,y) \mid y \in C\ \}.$$

The number

$$\rho(C) := \max\{\ d(x, C) \mid x \in V\Gamma\ \}$$

is called the *covering radius* of C.

For C a nonempty subset of $V\Gamma$ and for $0 \le i \le \rho$, define

$$C_i = \{ \; x \in V\Gamma \mid d(x, C) = i \; \}.$$

Then C_i is called the i^{th} *subconstituent* of C and $\Pi(C) = \{C_0 = C, C_1, \ldots, C_\rho\}$ is called the *distance partition* of $V\Gamma$ with respect to code C.

A partition $\Pi = \{P_0, P_1, \ldots, P_k\}$ of $V\Gamma$ is called *equitable* if, for all i and j, the number of neighbors a vertex in P_i has in P_j is independent of the choice of vertex in P_i. We say a code C in Γ is *completely regular* if this distance partition $\Pi(C)$ is equitable[1]. In this case the following quantities are well-defined:

$$\gamma_i = |\{y \in C_{i-1} \mid d(x, y) = 1\}| \,, \tag{3.1}$$

$$\alpha_i = |\{y \in C_i \mid d(x, y) = 1\}| \,, \tag{3.2}$$

$$\beta_i = |\{y \in C_{i+1} \mid d(x, y) = 1\}| \tag{3.3}$$

where x is chosen from C_i. The numbers $\gamma_i, \alpha_i, \beta_i$ are called the *intersection numbers* of code C. Observe that a graph Γ is distance-regular if and only if each vertex is a completely regular code and these $|V\Gamma|$ codes all have the same intersection numbers. An equitable partition $\Pi = \{P_1, \ldots, P_m\}$ of $V\Gamma$ is called a *completely regular partition* if all P_i are completely regular codes and any two of these have the same parameters.

If $\mathbf{x}$ is the characteristic vector of C as a subset of $V\Gamma$, then the *outer distribution module* of C is defined as

$$\mathbb{A}\mathbf{x} = \{M\mathbf{x} \mid M \in \mathbb{A}\}.$$

Clearly, this is an $\mathbb{A}$-invariant subspace of the standard module $V = \mathbb{R}^{V\Gamma}$. Our next goal is to describe two nice bases for $\mathbb{A}\mathbf{x}$.

For $0 \le i \le \rho$, let $\mathbf{x}_i$ denote the characteristic vector of C_i.

LEMMA 3.1. *Let Γ be a distance-regular graph and C a completely regular code in Γ. With notation as above, we have*

(a) *the vectors $\{\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\rho\}$ form a basis for the outer distribution module $\mathbb{A}\mathbf{x}$ of C;*

(b) *relative to this basis, the matrix representing the action of A on $\mathbb{A}\mathbf{x}$ is given by the tridiagonal matrix*

$$U := U(C) = \begin{pmatrix} \alpha_0 & \beta_0 & & & \\ \gamma_1 & \alpha_1 & \beta_1 & & \\ & \gamma_2 & \alpha_2 & \beta_2 & \\ & & \ddots & \ddots & \ddots \\ & & & \gamma_\rho & \alpha_\rho \end{pmatrix}; \tag{3.4}$$

(c) $\dim \mathbb{A}\mathbf{x} = \rho + 1$.

PROOF. From Equations (3.1), (3.2) and (3.3) above, we have

$$A\mathbf{x}_i = \beta_{i-1}\mathbf{x}_{i-1} + \alpha_i\mathbf{x}_i + \gamma_{i+1}\mathbf{x}_{i+1} \tag{3.5}$$

[1] This definition of a completely regular code is due to Neumaier [**13**]. When Γ is distance-regular, it is equivalent to the original definition, due to Delsarte [**6**], which we now mention. If $\mathbf{x}$ is the characteristic vector of C, construct a $|V\Gamma| \times (D+1)$ matrix with columns $A_i\mathbf{x}$ $(0 \le i \le D)$. Delsarte declares C to be completely regular if this *outer distribution matrix* has only $\rho+1$ distinct rows.

for $0 \le i \le \rho$ where, for convenience, we set $\mathbf{x}_{-1} = \mathbf{0}$ and $\mathbf{x}_{\rho+1} = \mathbf{0}$. So a simple inductive argument shows that each $\mathbf{x}_i$ lies in the outer distribution module of C. These vectors are trivially linearly independent, so we need only verify that they span $\mathbb{A}\mathbf{x}$. By (3.5), these vectors span an A-invariant subspace of V containing the characteristic vector $\mathbf{x}$ of C; since $\mathbb{A}\mathbf{x}$ is defined to be the smallest such subspace, the two spaces must coincide. □

COROLLARY 3.2. *Let Γ be a distance-regular graph. For any completely regular code C in Γ with characteristic vector $\mathbf{x}$, the outer distribution module $\mathbb{A}\mathbf{x}$ of C is closed under entrywise multiplication.*

PROOF. Simply observe that the basis vectors $\mathbf{x}_i$ satisfy $\mathbf{x}_i \circ \mathbf{x}_j = \delta_{i,j}\mathbf{x}_i$. □

The tridiagonal matrix U defined by (3.4) is called the *quotient matrix* of Γ with respect to C.

Now note that, for $0 \le j \le D$, if the the vector $E_j\mathbf{x}$ is not the zero vector, then it is an eigenvector for A with eigenvalue θ_j. This motivates us to define

$$S^*(C) = \{j \mid 1 \le j \le D,\ E_j\mathbf{x} \ne \mathbf{0}\}.$$

LEMMA 3.3. *Let Γ be a distance-regular graph and C a completely regular code in Γ. With notation as above, we have*

(a) *the nonzero vectors among the set $\{E_j\mathbf{x} \mid 0 \le j \le D\}$ form a basis for the outer distribution module $\mathbb{A}\mathbf{x}$ of C;*
(b) *relative to this basis, the matrix representing the action of A on $\mathbb{A}\mathbf{x}$ is a diagonal matrix with diagonal entries $\{\theta_j \mid j \in S^*(C) \cup \{0\}\}$;*
(c) $|S^*(C)| = \rho$.

PROOF. Since $\mathbb{A}$ is spanned both by $\{A_i\}_{i=0}^D$ and $\{E_i\}_{i=0}^D$, we see that $\mathbb{A}\mathbf{x}$ is spanned by both $\{A_i\mathbf{x}\}_{i=0}^D$ and $\{E_i\mathbf{x}\}_{i=0}^D$. Since the nonzero vectors in this latter set are linearly independent, they form a basis for $\mathbb{A}\mathbf{x}$. From Lemma 3.1(c), we see that there must be exactly $\rho+1$ nonzero vectors in this set, so $|S^*(C)| = \rho$. Finally, we have $AE_j\mathbf{x} = \theta_j E_j\mathbf{x}$ showing that the matrix representing the action of A on $\mathbb{A}\mathbf{x}$ relative to this basis is a diagonal matrix with diagonal entries as claimed. □

COROLLARY 3.4. *Let Γ be a distance-regular graph and let C be a completely regular code in Γ. With notation as above, the quotient matrix U has $\rho+1$ distinct eigenvalues, namely $\{\theta_j \mid j \in S^*(C) \cup \{0\}\}$.*

PROOF. Suppose $S^*(C) = \{i_1, \ldots, i_\rho\}$. Since both U and the diagonal matrix $\mathrm{diag}\,(k, \theta_{i_1}, \ldots, \theta_{i_\rho})$ represent the same linear transformation, A, on the module $\mathbb{A}\mathbf{x}$ with respect to different bases, these two matrices must have the same eigenvalues. □

For C a completely regular code in Γ, we say that η is an *eigenvalue of C* if η is an eigenvalue of the quotient matrix U defined in (3.4). By $\mathrm{Spec}\,(C)$, we denote the set of eigenvalues of C. The above corollary is often called "Lloyd's Theorem" in coding theory. The condition that each eigenvalue of C must be an eigenvalue of Γ is a powerful condition on the existence of completely regular codes, and perfect codes in particular[2].

[2] A code C in a distance-regular graph is *perfect* if $|C| = 1$ or $\delta(C) = 2\rho(C) + 1$. All perfect codes are completely regular [**6**, Theorem 5.13].

Note that, since $\gamma_i + \alpha_i + \beta_i = k$ for all i, $\theta_0 = k$ belongs to $\mathrm{Spec}\,(C)$. So

$$\mathrm{Spec}\,(C) = \{k\} \cup \{\theta_j \mid j \in S^*(C)\}\,.$$

Set $\mathrm{Spec}^*(C) := \mathrm{Spec}\,(C) - \{k\}$. For any eigenvalue η of C, there is a unique right eigenvector

$$u(\eta) := [u_0(\eta) = 1, u_1(\eta), \ldots, u_\rho(\eta)]^\top \tag{3.6}$$

of U associated to η; in analogy with the standard right eigenvectors of graph Γ, we refer to this vector as the *standard (right) eigenvector* of C belonging to η. Note that this vector satisfies the following initial conditions and recurrence relation:

$$\begin{gathered} u_0(\eta) = 1,\ \ u_1(\eta) = \frac{\eta - \alpha_0}{\beta_0}, \\ \gamma_i u_{i-1}(\eta) + \alpha_i u_i(\eta) + \beta_i u_{i+1}(\eta) = \eta u_i(\eta) \quad (0 \le i \le \rho), \\ \text{where } \ u_{-1}(\eta) = u_{\rho+1}(\eta) = 0 \end{gathered} \tag{3.7}$$

with (2.3) as a special case. As with the case where $|C| = 1$, for each standard right eigenvector of C, there is an eigenvector of Γ in $\mathbb{A}\mathbf{x}$ with the same eigenvalue which is unique up to scalar multiplication. For eigenvalue θ_j of C, we refer to this *eigenvector belonging to* C either as $E_j\mathbf{x}$ or as

$$\mathbf{u}(\theta_j) = \sum_{i=0}^{\rho} u_i(\theta_j)\mathbf{x}_i \tag{3.8}$$

where $u(\theta_j)$ is defined in (3.6) above, these two definitions differing only in their magnitude. Note that

$$\mathbf{u}(\theta_j) = \frac{|V\Gamma|}{m_j} E_j\mathbf{x} \in \mathbb{A}\mathbf{x} \cap V_j. \tag{3.9}$$

Lemma 3.5. *Assume that Γ is Q-polynomial with respect to the ordering $\theta_0 = k, \theta_1, \ldots, \theta_D$ of its eigenvalues. Let C be a completely regular code with* $\mathrm{Spec}^*(C) = \{\theta_{i_1}, \theta_{i_2}, \ldots, \theta_{i_\rho} \mid i_1 < i_2 < \cdots < i_\rho\}$. *If $E_{i_1}\mathbf{x}$ has $\rho + 1$ different entries, then $i_j - i_{j-1} \le i_1$ for all $j \in \{1, \ldots, \rho\}$.*

Proof. By Lemma 3.1(c), the outer distribution module $\mathbb{A}\mathbf{x}$ of C has dimension $\rho + 1$ and by Lemma 3.3(a), $\{E_{i_j}\mathbf{x} \mid i_j \in S^*(C) \cup \{0\}\}$ is a basis for it. We now consider the entrywise product $\mathbf{u}^{(\ell)}$ of ℓ copies of the vector $\mathbf{u} := E_{i_1}\mathbf{x}$. Note that $\mathbf{u}^{(\ell)} \in \mathbb{A}\mathbf{x}$ and that $\Lambda := \{\mathbf{u}^{(\ell)} \mid 0 \le \ell \le \rho\}$ is a linearly independent set of size $\rho + 1$ by the Vandermonde property. So Λ spans $\mathbb{A}\mathbf{x}$. Suppose that $i_h - i_{h-1} \le i_1$ for $h < j$ but $i_j > i_{j-1} + i_1$. Set

$$\mathcal{W}' = \mathrm{span}\left\{E_0\mathbf{x}, E_{i_1}\mathbf{x}, \ldots, E_{i_{j-1}}\mathbf{x}\right\}.$$

As $\mathbb{A}\mathbf{x}$ is closed under the Hadamard product, $\mathbf{u} \circ \mathcal{W}' \subseteq \mathbb{A}\mathbf{x}$. As $q^l_{i_1,h} = 0$ for $h \le i_{j-1}$ and $l \ge i_j$, we also have $\mathbf{u} \circ \mathcal{W}' \subseteq V_0 + V_{i_1} + \cdots + V_{i_{j-1}+i_1}$ using Theorem 2.2. Hence

$$\mathbf{u} \circ \mathcal{W}' \subseteq \mathbb{A}\mathbf{x} \cap (V_0 + V_{i_1} + \cdots + V_{i_{j-1}+i_1}).$$

That is, $\mathbf{u} \circ \mathcal{W}' \subseteq \mathcal{W}'$ and so $\mathbf{u}^{(\ell)} \circ \mathcal{W}' \subseteq \mathcal{W}'$ for $\ell \ge 1$ contradicting the fact that Λ spans $\mathbb{A}\mathbf{x}$. $\square$

COROLLARY 3.6. *Let Γ be a distance-regular graph and assume Γ is Q-polynomial with respect to the natural ordering $\theta_0 = k > \theta_1 > \cdots > \theta_D$ of its eigenvalues. Let C be a completely regular code in Γ with $S^*(C) = \{i_1, \ldots, i_\rho\}$ where $i_1 < \cdots < i_\rho$ and $\rho = \rho(C)$. Then $i_j - i_{j-1} \le i_1$ for all $j \in \{1, \ldots, \rho\}$.*

PROOF. A standard argument involving Sturm sequences (see, e.g., [**2**, p. 130] and [**7**, Lemma 8.5.2]) shows that, if θ_{i_1} is the second largest eigenvalue of the tridiagonal matrix U in (3.4), then the entries of the standard right eigenvector of C belonging to θ_{i_1} are strictly decreasing. So the eigenvector $\mathbf{u}(\theta_{i_1})$ has $\rho + 1$ distinct entries as required. □

Our computational work suggests that Corollary 3.6 is often a strong feasibility condition for completely regular codes in the Hamming graphs.

Let Γ be a distance-regular graph with diameter $D \ge 2$. We say Γ is an antipodal 2-cover whenever for all $x \in V\Gamma$, there exists a unique vertex $y \in V\Gamma$ such that $d(x, y) = D$. We denote this vertex by $\pi(x)$ and note that the mapping $\pi : V\Gamma \longrightarrow V\Gamma$ is an automorphism of Γ. It is known (cf. [**2**, Prop. 4.2.3(ii)]) that the subspace stabilized by this mapping is

$$\left\{\mathbf{v} \in V \;\mid\; \mathbf{v}_x = \mathbf{v}_{\pi(x)} \;\forall (x \in V\Gamma)\right\} = V_0 + V_2 + \cdots + V_{2\lfloor \frac{D}{2} \rfloor}$$

and is therefore an $\mathbb{A}$-submodule of the standard module.

LEMMA 3.7. *Let Γ be an antipodal 2-cover distance-regular graph and let $\theta_0 > \theta_1 > \cdots > \theta_D$ be the distinct eigenvalues of Γ. Let C be a completely regular code with $S^*(C) = \{i_0 = 0 < i_1 < \ldots < i_\rho\}$ where $\rho = \rho(C)$. Let π be the automorphism defined above. Then either*

$$\pi(C) = C \;\; \textit{and} \;\; i_j \equiv 0 \pmod 2 \;\forall (j \in \{0, \ldots, \rho\})$$

or

$$\pi(C) = C_\rho \;\; \textit{and} \;\; i_j \equiv j \pmod 2 \;\forall (j \in \{0, \ldots, \rho\}).$$

PROOF. We know that $\mathbb{A}\mathbf{x}$ is invariant under any A_i. So

$$A_D\mathbf{x} = \tau_0\mathbf{x}_0 + \cdots + \tau_\rho\mathbf{x}_\rho$$

for some scalars $\tau_0, \ldots, \tau_\rho$. Let $x \in C$ and assume $\pi(x) \in C_i$ for some i. Then $\tau_i \neq 0$ and so for any vertex $y \in C_i$, $|\{z \in C \;:\; d(y, z) = D\}| = 1$. This gives $C_i \subseteq \pi(C)$. Since $\rho(\pi(C)) = \rho(C)$, the code $\pi(C)$ is either C or C_ρ.

Let us first consider the case where $\pi(C) = C$. In this case, the characteristic vector of C belongs to the A-submodule $V_0 + V_2 + \cdots$ as outlined above, so for each j, $E_{i_j}\mathbf{x}$ belongs to this submodule as well. Thus $i_j \equiv 0 \bmod 2$ for all $0 \le j \le \rho$.

In the other case, $\pi(C) = C_\rho$ and we use a Sturm sequence argument. We know that

$$E_{i_j}\mathbf{x} = \frac{m_{i_j}}{|V\Gamma|}\left(u_0\mathbf{x} + u_1\mathbf{x}_1 + \cdots + u_\rho\mathbf{x}_\rho\right)$$

where $[u_0, u_1, \ldots, u_\rho]^\top$ is the standard eigenvector of C belonging to eigenvalue θ_{i_j}. But, by hypothesis, θ_{i_j} is the j^{th} largest eigenvalue of the tridiagonal quotient matrix U defined in (3.4). So by [**7**, Lemma 8.5.2], the sequence $u_0, u_1, \ldots, u_\rho$ has j sign changes. Since $u_0 > 0$, we find u_ρ is positive for j even and negative for j odd. But it is well-known that if $\mathbf{v}$ is an eigenvector of an antipodal 2-cover Γ, $\mathbf{v} \in V_i$, then $\mathbf{v}_{\pi(x)} = \mathbf{v}_x$ for each $x \in V\Gamma$ when i is even and $\mathbf{v}_{\pi(x)} = -\mathbf{v}_x$ for each $x \in V\Gamma$ when i is odd. From this we obtain our result. □

4. Leonard completely regular codes and the Q-polynomial property

In this section, we will define Leonard completely regular codes, investigate their connection to Q-polynomial distance-regular coset graphs, and show that the intersection array and eigenvalues of a Leonard completely regular code are all determined by just a few paramters.

4.1. Leonard codes and coset graphs. Let Γ be a distance-regular graph with diameter D and $\mathrm{Spec}\,(\Gamma) = \{\theta_0, \ldots, \theta_D\}$. Let C be a completely regular code with covering radius ρ in Γ.

DEFINITION 4.1. A completely regular code C in Γ is said to be *Leonard* if we have an ordering $\mathrm{Spec}\,(C) = \{\theta_0, \theta_{i_1}, \ldots, \theta_{i_\rho}\}$ of the eigenvalues of C such that the following hold:

$$\mathbf{u}^{(\ell)} := \underbrace{\mathbf{u} \circ \mathbf{u} \circ \cdots \circ \mathbf{u}}_{\ell \text{ times}} \in V_{i_0} + V_{i_1} + \cdots + V_{i_\ell} \setminus (V_{i_0} + V_{i_1} + \cdots + V_{i_{\ell-1}}),$$

where $\mathbf{u} = E_{i_1}\mathbf{x}$.

In this case, we say C is *Leonard with respect to the ordering* $\theta_0, \theta_{i_1}, \ldots, \theta_{i_\rho}$.

REMARK 4.2. If C is completely regular, then it follows that $\{\mathbf{u}^{(i)} \mid i = 0, 1, \ldots, \rho\}$ are linearly independent, where $\mathbf{u} = E_{i_1}\mathbf{x}$. It follows from the theory of Vandermonde matrices that this is equivalent with the fact that $\mathbf{u}$ has exactly $\rho + 1$ distinct entries.

REMARK 4.3. Let Γ be a distance-regular graph and $x \in V\Gamma$. Then $C = \{x\}$ is completely regular and C is Leonard with respect to the ordering $\theta_0, \theta_{i_1}, \ldots, \theta_{i_D}$ of $\mathrm{Spec}\,(C)$ if and only if Γ is Q-polynomial with respect to the ordering $E_0, E_{i_1}, \ldots, E_{i_D}$ of its primitive idempotents.

LEMMA 4.4. *Let C be a completely regular code in a distance-regular graph Γ and assume that C is Leonard with respect to the ordering $\theta_0, \theta_{i_1}, \ldots, \theta_{i_\rho}$ of $\mathrm{Spec}\,(C)$. For $1 \leq j \leq \rho$, we have $E_{i_j}\mathbf{u}^{(j)} \neq \mathbf{0}$ and $E_{i_\ell}\mathbf{u}^{(j)} = \mathbf{0}$ for $j < \ell \leq \rho$.*

PROOF. Similar to the proof of Lemma 3.5. □

Note that any completely regular code with covering radius at most two is Leonard. In the Hamming graphs there are many such codes, namely let C be a linear code with exactly two non-zero weights. Then the dual of C is a completely regular code with covering radius two. In Calderbank and Kantor [**4**] a survey on the constructions of linear codes with exactly two non-zero weights is given.

Also if we take for C an antipodal pair in a doubled Odd graph Γ (see, for example [**2**, Sec. 9.1D]) then C is Leonard but Γ is not Q-polynomial if its valency is at least 3.

Let X be a finite abelian group. A *translation distance-regular graph* on X is a distance-regular graph Γ with vertex set X such that if x and y are adjacent then $x + z$ and $y + z$ are adjacent for all $x, y, z \in X$. A code $C \subseteq X$ is called *additive* for all $x, y \in C$, also $x - y \in C$; i.e., C is a subgroup of X. If C is an additive code in a translation distance-regular graph on X, then we obtain the usual coset partition $\Delta(C) := \{C + x \mid x \in X\}$ of X; whenever C is a completely regular code, it is easy to see that $\Delta(C)$ is a completely regular partition. For any additive code C in a translation distance-regular graph Γ on vertex set X, the *coset graph* of C in Γ is the graph with vertex set X/C and an edge joining coset C' to coset C''

whenever Γ has an edge with one end in C' and the other in C''. It follows from Theorem 11.1.6 in [**2**] that this coset graph is distance-regular whenever C is an additive completely regular code in a translation distance-regular graph.

PROPOSITION 4.5. *Let X be a finite abelian group and let Γ be a translation distance-regular graph on X. Let C be an additive completely regular code in Γ and let $\Delta(C)$ be the partition of X into cosets of C. Then C is Leonard if and only if $\Gamma/\Delta(C)$ is a Q-polynomial distance-regular graph.*

PROOF. Let C be an additive completely regular code in Γ whose intersection numbers are γ_i, α_i and β_i $(0 \le i \le \rho)$. Then by [**2**, p. 352-3], eigenvalues of $\Gamma/\Delta(C)$ are $\frac{\eta_i - \alpha_0}{\gamma_1}$ for $\eta_i \in \mathrm{Spec}\,(C)$. We see that $L(\Gamma/\Delta(C)) = \frac{1}{\gamma_1}(U - \alpha_0 I)$. Now the result follows easily. $\square$

4.2. A recurrence relation for the parameters. We now derive an important tool which will later allow us to find all of the parameters of a Leonard completely regular code from just a few parameters.

DEFINITION 4.6. Let C be a completely regular code in a distance-regular graph Γ and let η be an eigenvalue of C. Let $u(\eta) = [u_0(\eta) = 1, \ldots, u_\rho(\eta)]^\top$ be the standard eigenvector of C belonging to η. Then the η is called *non-degenerate* (for C) if $u_{i-1}(\eta) \neq u_i(\eta)$ $(1 \le i \le \rho)$ and $u_{i-1}(\eta) \neq u_{i+1}(\eta)$ $(1 \le i \le \rho - 1)$.

Note that the second largest eigenvalue of a completely regular code is always non-degenerate for that code. Likewise, if a code C is Leonard with respect to the ordering $\{\eta_0, \eta_1, \ldots, \eta_\rho\}$ of its eigenvalues, then η_1, by Remark 4.2, is non-degenerate for C.

PROPOSITION 4.7. *Let Γ be a distance-regular graph with valency k. Let C be a completely regular code with covering radius ρ and $\mathrm{Spec}\,(C) = \{\eta_i \mid 0 \le i \le \rho\}$ in Γ. Let $u(\eta_i) = [u_0 = 1, u_1(\eta_i), \ldots, u_\rho(\eta_i)]^T$ be the standard right eigenvector of C belonging to eigenvalue η_i, $(0 \le i \le \rho)$. Then there are (unique) $\lambda_i, \tau_i \in \mathbb{R}$ such that $\sum_i \lambda_i = 1, \sum_i \tau_i = 1$ and the following two hold:*

$$u^{(2)}(\eta_1) = \sum_{i=0}^{\rho} \lambda_i u(\eta_i) \tag{4.1}$$

and

$$u^{(3)}(\eta_1) = \sum_{i=0}^{\rho} \tau_i u(\eta_i) \tag{4.2}$$

In particular, if η_1 is non-degenerate then the intersection numbers of C are determined by the set of values

$$\{\eta_0, \eta_1\} \cup \{\eta_i \mid \lambda_i \neq 0 \ \ or \ \ \tau_i \neq 0\} \cup \{\lambda_0, \ldots, \lambda_\rho\} \cup \{\tau_0, \ldots, \tau_\rho\}\,.$$

PROOF. Let $u(\eta_i)$ be the standard eigenvector of C belonging to η_i. The set $\{u(\eta_0), \ldots, u(\eta_\rho)\}$ forms a basis of $\mathbb{R}^{\rho+1}$. Hence scalars λ_i and τ_i, each summing to one and satisfying (4.1) and (4.2), exist.

As $\gamma_j u_{j-1}(\eta_i) + \alpha_j u_j(\eta_i) + \beta_j u_{j+1}(\eta_i) = \eta_i u_j(\eta_i)$, (4.1) and (4.2) can be rewritten as

$$\gamma_j u_{j-1}^2(\eta_1) + \alpha_j u_j^2(\eta_1) + \beta_j u_{j+1}^2(\eta_1) = \sum_{i=0}^{\rho} \lambda_i \eta_i u_j(\eta_i)$$

and

$$\gamma_j u_{j-1}^3(\eta_1) + \alpha_j u_j^3(\eta_1) + \beta_j u_{j+1}^3(\eta_1) = \sum_{i=0}^{\rho} \tau_i \eta_i u_j(\eta_i).$$

Assume that we know the set $\{\eta_i \mid \lambda_i \neq 0 \text{ or } \tau_i \neq 0 \text{ or } i = 0,1\}$ and all the λ_i and τ_i. We use induction on j to recover $\gamma_j, \alpha_j, \beta_j$ as well as $u_{j+1}(\eta_i)$ for $1 \leq i \leq \rho$. For $j = 0$, the equations

$$\alpha_0 + \beta_0 = k,$$
$$\alpha_0 + \beta_0 u_1(\eta_i) = \eta_i \quad \text{for } 0 \leq i \leq \rho$$

and

$$\alpha_0 + \beta_0 u_1^2(\eta_1) = \sum_{i=0}^{\rho} \lambda_i \eta_i.$$

easily allow us to obtain[3] $\alpha_0, \beta_0, u_1(\eta_i)$ for $0 \leq i \leq \rho$. Suppose that, for all $j \leq m$, the numbers $\gamma_j, \alpha_j, \beta_j$, and $u_{j+1}(\eta_i)$ $(0 \leq i \leq \rho)$ are known. Now consider the case $j = m+1$; we have four equations:

$$\gamma_{m+1} + \alpha_{m+1} + \beta_{m+1} = k, \tag{4.3}$$

$$\gamma_{m+1} u_m(\eta_1) + \alpha_{m+1} u_{m+1}(\eta_1) + \beta_{m+1} u_{m+2}(\eta_1) = \eta_1 u_{m+1}(\eta_1), \tag{4.4}$$

$$\gamma_{m+1} u_m^2(\eta_1) + \alpha_{m+1} u_{m+1}^2(\eta_1) + \beta_{m+1} u_{m+2}^2(\eta_1) = \sum_{i=0}^{\rho} \lambda_i \eta_i u_{m+1}(\eta_i) \tag{4.5}$$

and

$$\gamma_{m+1} u_m^3(\eta_1) + \alpha_{m+1} u_{m+1}^3(\eta_1) + \beta_{m+1} u_{m+2}^3(\eta_1) = \sum_{i=0}^{\rho} \tau_i \eta_i u_{m+1}(\eta_i). \tag{4.6}$$

As η_1 is non-degenerate, we obtain by Equations (4.3)–(4.6):

$$\begin{aligned}
u_{m+2}(\eta_1) &= \frac{R_\tau - R_\lambda\left(u_{m+1}(\eta_1) + u_m(\eta_1)\right) + \eta_1 u_{m+1}^2(\eta_1) u_m(\eta_1)}{R_\lambda + k u_{m+1}(\eta_1) u_m(\eta_1) - \eta_1 u_{m+1}(\eta_1)\left(u_{m+1}(\eta_1) + u_m(\eta_1)\right)}, \\
\gamma_{m+1} &= \frac{R_\lambda + k u_{m+2}(\eta_1) u_{m+1}(\eta_1) - \eta_1 u_{m+1}(\eta_1)\left(u_{m+2}(\eta_1) + u_{m+1}(\eta_1)\right)}{(u_m(\eta_1) - u_{m+2}(\eta_1))(u_m(\eta_1) - u_{m+1}(\eta_1))}, \\
\alpha_{m+1} &= \frac{R_\lambda + k u_{m+2}(\eta_1) u_m(\eta_1) - \eta_1 u_{m+1}(\eta_1)\left(u_{m+2}(\eta_1) + u_m(\eta_1)\right)}{(u_{m+1}(\eta_1) - u_{m+2}(\eta_1))(u_{m+1}(\eta_1) - u_m(\eta_1))}, \\
\beta_{m+1} &= \frac{R_\lambda + k u_{m+1}(\eta_1) u_m(\eta_1) - \eta_1 u_{m+1}(\eta_1)\left(u_{m+1}(\eta_1) + u_m(\eta_1)\right)}{(u_{m+2}(\eta_1) - u_{m+1}(\eta_1))(u_{m+2}(\eta_1) - u_m(\eta_1))},
\end{aligned}$$

where R_λ and R_τ are shorthand for the expressions on the right-hand sides of Equations (4.5) and (4.6), respectively; these quantities are presumed known by the induction hypothesis.

But we also have, for $0 \leq i \leq \rho$,

$$\gamma_{m+1} u_m(\eta_i) + \alpha_{m+1} u_{m+1}(\eta_i) + \beta_{m+1} u_{m+2}(\eta_i) = \eta_i u_{m+1}(\eta_i) \tag{4.7}$$

[3]Indeed, $\beta_0 \neq 0$. If we denote by S the sum on the right-hand side of the last equation, the simultaneous equations $k + \beta_0(u_1(\eta_1) - 1) = \eta_1$ and $k + \beta_0(u_1(\eta_1)^2 - 1) = S$ allow us to solve for $u_1(\eta_1) + 1$ and then for β_0 so that all the remaining equations become linear.

with (4.3) and (4.4) as special cases; from these, we now obtain $u_{m+2}(\eta_i)$ for $2 \leq i \leq \rho$. $\square$

In the statement of Proposition 4.7 above, there can be as many as $D+1$ nonzero λ_i and as many as $D+1$ nonzero τ_i. We now observe that, when the graph Γ has many vanishing Krein parameters, these numbers tend to be much smaller.

LEMMA 4.8. *Let λ_j and τ_j be the constants defined in Proposition 4.7 above. Suppose that* $\mathrm{Spec}^*(C) = \{\theta_{i_1}, \ldots, \theta_{i_\rho}\}$*. If $\lambda_j \neq 0$, then $q^{i_j}_{i_1,i_1} \neq 0$ and if $\tau_j \neq 0$, then there exists i_ℓ such that $q^{i_\ell}_{i_1,i_1} \neq 0$ and $q^{i_j}_{i_\ell,i_1} \neq 0$.*

PROOF. Recall the vectors $\mathbf{u}(\theta_{i_j}) = \sum_{h=0}^{\rho} u_h(\theta_{i_j})\mathbf{x}_h$ defined in (3.8). From (4.1) and (4.2) respectively, we have

$$\mathbf{u}^{(2)}(\theta_{i_1}) = \sum_{j=0}^{\rho} \lambda_j \mathbf{u}(\theta_{i_j}), \qquad \mathbf{u}^{(3)}(\theta_{i_1}) = \sum_{j=0}^{\rho} \tau_j \mathbf{u}(\theta_{i_j}).$$

Since $\mathbf{u}(\theta_{i_j}) \in V_{i_j}$, if $\lambda_j \neq 0$, then $\mathbf{u}^{(2)}(\theta_{i_1})$ is not orthogonal to V_{i_j}. So, by Theorem 2.2, $q^{i_j}_{i_1,i_1} \neq 0$. Likewise, since $\mathbf{u}^{(3)}(\theta_{i_1}) = \sum_{\ell=0}^{\rho} \lambda_\ell \mathbf{u}(\theta_{i_\ell}) \circ \mathbf{u}(\theta_{i_1})$, if $\tau_j \neq 0$ then there exists ℓ such that $\lambda_\ell \neq 0$ and $\mathbf{u}(\theta_{i_\ell}) \circ \mathbf{u}(\theta_{i_1})$ is not orthogonal to V_{i_j}. This time, Theorem 2.2 implies that there exists i_ℓ such that $q^{i_\ell}_{i_1,i_1} \neq 0$ and $q^{i_j}_{i_\ell,i_1} \neq 0$. $\square$

5. Leonard codes and Leonard pairs

In the previous section, we defined a Leonard completely regular code in terms of entrywise products of the eigenvectors of the code (Definition 4.1). Our goal in this section is to show that a code is Leonard if and only if a certain pair of linear transformations form a Leonard pair on its outer distribution module.

Let Γ be a distance-regular graph with adjacency matrix A and let $C \subseteq V\Gamma$ be a completely regular code with covering radius ρ, $\mathrm{Spec}^*(C) = \{\theta_{i_1}, \ldots, \theta_{i_\rho}\}$ and distance partition $\{C_0, C_1, \ldots, C_\rho\}$. For $0 \leq i \leq \rho$, let $\mathbf{x}_i$ denote the characteristic vector of subconstituent C_i, and abbreviate $\mathbf{x}_0 = \mathbf{x}$. Let $\mathcal{B}^* := \{\mathbf{x}_i \mid i = 0, \ldots, \rho\}$ and $\mathcal{B} := \{E_{i_j}\mathbf{x} \mid j = 0, \ldots, \rho\}$. Then both $\mathcal{B}^*$ and $\mathcal{B}$ are bases for the outer distribution module $\mathbb{A}\mathbf{x}$ of C (by Lemma 3.1 and Lemma 3.3, respectively). Now consider first the linear transformation $\mathcal{A}$ on $\mathbb{A}\mathbf{x}$ which is defined by

$$\mathcal{A}(\mathbf{y}) = A\mathbf{y} \quad \text{for} \quad \mathbf{y} \in \mathbb{A}\mathbf{x}\ . \tag{5.1}$$

Since $A\mathbf{x}_i = \beta_{i-1}\mathbf{x}_{i-1} + \alpha_i\mathbf{x}_i + \gamma_{i+1}\mathbf{x}_{i+1}$ by (3.5), the matrix representing $\mathcal{A}$ with respect to the basis $\mathcal{B}^*$ is irreducible tridiagonal (i.e., a tridiagonal matrix with all entries in the superdiagonal and subdiagonal nonzero) and the matrix representing $\mathcal{A}$ with respect to the basis $\mathcal{B}$ is diagonal.

Next, for any nontrivial eigenvalue θ of C, define the linear transformation $\mathcal{A}^*(\theta)$ on $\mathbb{A}\mathbf{x}$ by

$$\mathcal{A}^*(\theta)(\mathbf{y}) = (E(\theta)\mathbf{x}) \circ \mathbf{y} \quad \text{for} \quad \mathbf{y} \in \mathbb{A}\mathbf{x}\ . \tag{5.2}$$

We can easily check that the matrix representing $\mathcal{A}^*(\theta)$ with respect to the basis $\mathcal{B}^*$ is diagonal as $(E(\theta)\mathbf{x}) \circ \mathbf{x}_i = \frac{m}{|V\Gamma|} u_i(\theta)\mathbf{x}_i$ where $m = \operatorname{rank} E(\theta)$ using (3.9). But, in general, one cannot say much about the matrix representing $\mathcal{A}^*(\theta)$ with respect to basis $\mathcal{B}^*$.

5.1. Leonard pairs. The structure of the linear transformations $\mathcal{A}$ and $\mathcal{A}^*$ defined above motivates us to consider Terwilliger's concept of a Leonard pair, which we now define.

DEFINITION 5.1 ([**17**, p. 150]). Let V be a vector space of finite positive dimension and let $\mathcal{A}$ and $\mathcal{A}^*$ be two linear transformations on V that satisfy the following two conditions:

(a) there is a basis for V with respect to which the matrix representing $\mathcal{A}$ is irreducible tridiagonal and the matrix representing $\mathcal{A}^*$ is diagonal;
(b) there is a basis for V with respect to which the matrix representing $\mathcal{A}^*$ is irreducible tridiagonal and the matrix representing $\mathcal{A}$ is diagonal.

Then the pair $\mathcal{A}, \mathcal{A}^*$ is called a *Leonard pair* on V.

Leonard pairs were introduced by Terwilliger [**17**] in the study of P- and Q-polynomial association schemes but have since been shown to have far wider applicability. We refer the reader to [**20**] for a recent survey of the literature on this rapidly expanding topic. In fact, the Leonard pairs have been completely classified, and a parametrization is given by Terwilliger in [**19**]. We will employ this parametrization in Section 5.3 below. For now, we need one important result from [**18**]. Recall that a linear transformation from a finite-dimensional complex vector space V to itself is *multiplicity-free* if it has $\dim V$ distinct eigenvalues in $\mathbb{C}$. Of course, any such transformation is diagonalizable over $\mathbb{C}$ and its primitive idempotents all have rank one.

LEMMA 5.2 (Terwilliger [**18**, Lemma 5.7, Lemma 1.4]). *Let V be a vector space of finite positive dimension $d+1$ and let*

$$\mathcal{A} : V \to V, \qquad \mathcal{A}^* : V \to V$$

be multiplicity-free linear transformations on V. Let $F_0, F_1, \ldots, F_d$ be an ordering of the primitive idempotents of $\mathcal{A}$ and let $F_0^*, F_1^*, \ldots, F_d^*$ be an ordering of the primitive idempotents of $\mathcal{A}^*$. Consider the four conditions

$$F_h^* \mathcal{A} F_j^* \begin{cases} = 0 & \text{if } h-j>1 \\ \neq 0 & \text{if } h-j=1 \end{cases} \qquad (0 \le h,j \le d), \tag{5.3}$$

$$F_h^* \mathcal{A} F_j^* \begin{cases} = 0 & \text{if } j-h>1 \\ \neq 0 & \text{if } j-h=1 \end{cases} \qquad (0 \le h,j \le d), \tag{5.4}$$

$$F_h \mathcal{A}^* F_j \begin{cases} = 0 & \text{if } h-j>1 \\ \neq 0 & \text{if } h-j=1 \end{cases} \qquad (0 \le h,j \le d), \tag{5.5}$$

$$F_h \mathcal{A}^* F_j \begin{cases} = 0 & \text{if } j-h>1 \\ \neq 0 & \text{if } j-h=1 \end{cases} \qquad (0 \le h,j \le d). \tag{5.6}$$

If any three of (5.3)–(5.6), then each of (5.3)–(5.6) holds and $\mathcal{A}, \mathcal{A}^*$ is a Leonard pair on V. Conversely, if $\mathcal{A}, \mathcal{A}^*$ is a Leonard pair on V then there exist orderings $F_0, \ldots, F_d$ and $F_0^*, \ldots, F_d^*$ of the primitive idempotents of $\mathcal{A}$ and $\mathcal{A}^*$, respectively, for which each of (5.3)–(5.6) holds.

REMARK 5.3. Note that we are working only over the complex field and V is irreducible as a $\mathrm{Hom}_{\mathbb{C}}(V,V)$-module, so the conditions of [**18**, Lemma 5.7] apply.

5.2. Establishing equivalence. We now prove the main result of the paper.

THEOREM 5.4. *Let C be a completely regular code in a distance-regular graph Γ of valency k.*

(a) *If C is Leonard with respect to the ordering $\theta_{i_0} = k, \theta_{i_1}, \ldots, \theta_{i_\rho}$ of* Spec (C)*, then the transformations $\mathcal{A}$ and $\mathcal{A}^*(\theta_{i_1})$ defined in (5.1) and (5.2), respectively, form a Leonard pair on the outer distribution module $\mathbb{A}\mathbf{x}$ of C.*

(b) *Conversely, if, for some eigenvalue θ of C the matrices $\mathcal{A}$ and $\mathcal{A}^*(\theta)$ defined in (5.1) and (5.2), respectively, form a Leonard pair on the outer distribution module $\mathbb{A}\mathbf{x}$ of C, then C is Leonard with respect to some ordering $\theta_{i_0}, \theta_{i_1}, \ldots, \theta_{i_\rho}$ of* Spec (C) *in which $\theta_{i_0} = k$ and $\theta_{i_1} = \theta$.*

PROOF. Assume first that C is a completely regular code with covering radius ρ in Γ which is Leonard with respect to the ordering $\theta_{i_0}, \theta_{i_1}, \ldots, \theta_{i\rho}$ of Spec (C) where $\theta_{i_0} = k$. Let $\mathcal{A}$ and $\mathcal{A}^* = \mathcal{A}(\theta_{i_1})$ be the transformations defined in (5.1) and (5.2), respectively. Then, with $\mathbf{u} = E_{i_1}\mathbf{x}$,

$$\mathcal{A}^*\mathbf{y} = \mathbf{u} \circ \mathbf{y}, \qquad (\mathbf{y} \in \mathbb{A}\mathbf{x}).$$

We can order the eigenspaces of $\mathcal{A}$ and $\mathcal{A}^*$ so that the primitive idempotents of $\mathcal{A}$ are

$$F_j = \xi_j \mathbf{u}_j \mathbf{u}_j^\top, \qquad (0 \le j \le \rho)$$

where $\mathbf{u}_j := E_{i_j}\mathbf{x}$ and $\xi_j = \left(\mathbf{x}^\top E_{i_j}\mathbf{x}\right)^{-1}$ is a positive scalar. Dually, the primitive idempotents of $\mathcal{A}^*$ are

$$F_j^* = \xi_j^* \mathbf{x}_j \mathbf{x}_j^\top, \qquad (0 \le j \le \rho)$$

where $\xi_j^* = 1/|C_j|$.

It is straightforward to verify that statements (5.3) and (5.4) hold for $\mathcal{A}$. Using Lemma 5.2, it suffices now to verify that (5.5) holds for $\mathcal{A}^*$.

In order to do this, write $\mathbf{y} \in \mathbb{A}\mathbf{x}$ as $\mathbf{y} = \sum_{\ell=0}^{\rho} \eta_\ell \mathbf{u}_\ell$. For $0 \le h, j \le \rho$, we have

$$\begin{aligned} F_h \mathcal{A}^* F_j \mathbf{y} &= \xi_h \xi_j \mathbf{u}_h \mathbf{u}_h^\top \left(\mathbf{u} \circ \left(\mathbf{u}_j \mathbf{u}_j^\top \mathbf{y}\right)\right) \\ &= \sum_\ell \eta_\ell \xi_h \xi_j \mathbf{u}_h \mathbf{u}_h^\top \left(\mathbf{u} \circ \left(\mathbf{u}_j \mathbf{u}_j^\top \mathbf{u}_\ell\right)\right) \\ &= \eta_j \xi_h \xi_j^2 \mathbf{u}_h \mathbf{u}_h^\top \left(\mathbf{u} \circ \mathbf{u}_j\right) \\ &= \tilde{q}_{1,j}^h \eta_j \xi_h^2 \xi_j^2 \mathbf{u}_h \end{aligned}$$

where we have expressed $\mathbf{u} \circ \mathbf{u}_j \in \mathbb{A}\mathbf{x}$ as

$$\mathbf{u} \circ \mathbf{u}_j = \sum_{\ell=0}^{\rho} \tilde{q}_{1,j}^\ell \mathbf{u}_\ell.$$

Now it follows from Definition 4.1 that $\tilde{q}_{1,j}^\ell = 0$ for $\ell > j+1$ and it follows from Lemma 4.4 that $\tilde{q}_{1,j}^{j+1} \neq 0$. So we have

$$F_h \mathcal{A}^* F_j \begin{cases} = 0 & \text{if } h - j > 1 \\ \neq 0 & \text{if } h - j = 1 \end{cases} \qquad (0 \le h, j \le d),$$

as required.

For the converse, assume that θ is an eigenvalue of C such that $\mathcal{A}, \mathcal{A}^*(\theta)$ is a Leonard pair on $\mathbb{A}\mathbf{x}$. Since $\mathcal{A}$ is multiplicity-free on $\mathbb{A}\mathbf{x}$, the matrix representing it with respect to an ordered basis is diagonal if and only if this ordered basis is an ordering and scaling of the basis $\mathcal{B}$. Let $E_{i_0}\mathbf{x}, E_{i_1}\mathbf{x}, \ldots, E_{i_\rho}\mathbf{x}$ be an ordering of this basis with respect to which the matrix, M say, representing $\mathcal{A}^* = \mathcal{A}^*(\theta)$ is irreducible tridiagonal. Choose j so that $\theta = \theta_j$; since $\mathcal{A}^*(E_0\mathbf{x}) \in V_j$ and M is irreducible tridiagonal, we must either have $i_0 = 0$ and $i_1 = j$ or $i_\rho = 0$ and $i_{\rho-1} = j$. We may assume without loss of generality that $i_0 = 0$ and $i_1 = j$. Since $\mathcal{A}^*$ is multiplicity-free, we see that $\mathbf{u} = E_{i_1}\mathbf{x}$ has $\rho + 1$ distinct entries. Now since M is irreducible tridiagonal, for $0 \leq j \leq \rho$ we have

$$\mathbf{u} \circ \mathbf{u}_j = \epsilon_j \mathbf{u}_{j-1} + \varphi_j \mathbf{u}_j + \psi_j \mathbf{u}_{j+1}$$

for some scalars ϵ_j, φ_j, ψ_j (ϵ_j and ψ_j being nonzero) where $\mathbf{u}_{-1} = \mathbf{u}_{\rho+1} = \mathbf{0}$. Now we see by induction that

$$\mathbf{u}^{(\ell)} \in V_{i_0} + V_{i_1} + \cdots + V_{i_\ell}. \tag{5.7}$$

for $1 \leq \ell \leq \rho$ and we have that C is Leonard with respect to the ordering $\theta_{i_0}, \theta_{i_1}, \ldots, \theta_{i_\rho}$ of $\mathrm{Spec}\,(C)$. □

5.3. The classification problem for Leonard completely regular codes. In [**19**], Terwilliger gave a parametrization of any Leonard pair. It follows that, for any Leonard pair, there are at most seven free parameters. (Allowing for equivalence under affine transformations, this may be reduced to five.) We now show that the Leonard pair associated to a Leonard completely regular code in a known distance-regular graph has all its parameters determined by just six free parameters (as α_0 is usually non-zero).

Corollary 5.5. *Let Γ be a distance-regular graph of valency k and diameter D. Let C be a completely regular code in Γ which is Leonard with respect to the ordering $\eta_0, \eta_1, \ldots, \eta_\rho$ of* $\mathrm{Spec}\,(C)$. *Then the intersection numbers $\alpha_i, \beta_i, \gamma_i$ ($0 \leq i \leq \rho$) are completely determined (as is the covering radius ρ, from $\beta_\rho = 0$) by the eigenvalues η_1 and η_2 of C together with the parameters λ_0, λ_1, τ_1 and τ_2 as defined in Proposition 4.7.*

Proof. We again use the correspondence (3.8) between $\mathbb{A}\mathbf{x}$ and $\mathbb{R}^{\rho+1}$. Since C is Leonard with respect to the ordering $\eta_0, \eta_1, \ldots, \eta_\rho$, its standard right eigenvectors satisfy

$$u^{(2)}(\eta_1) = \lambda_0 u(\eta_0) + \lambda_1 u(\eta_1) + \lambda_2 u(\eta_2) \tag{5.8}$$

and

$$u^{(3)}(\eta_1) = \tau_0 u(\eta_0) + \tau_1 u(\eta_1) + \tau_2 u(\eta_2) + \tau_3 u(\eta_3). \tag{5.9}$$

with λ_i and τ_i as defined in Proposition 4.7. Looking at the zero entry on both sides of each equation, we find $\lambda_0 + \lambda_1 + \lambda_2 = 1$ and $\tau_0 + \tau_1 + \tau_2 + \tau_3 = 1$. Now consider $\mathcal{A}^*(\eta_1)$; this matrix is irreducible tridiagonal by Theorem 5.4, so there exist scalars $\sigma_1, \sigma_2, \sigma_3$ for which

$$u(\eta_1) \circ u(\eta_2) = \sigma_1 u(\eta_1) + \sigma_2 u(\eta_2) + \sigma_3 u(\eta_3). \tag{5.10}$$

Moreover, we have $\sigma_1 + \sigma_2 + \sigma_3 = 1$. Next, we may use this and Equation (5.8) to obtain an alternative expression for $u^{(3)}(\eta_1)$:

$$u^{(3)}(\eta_1) = \lambda_0\lambda_1 u(\eta_0) + \left(\lambda_0 + \lambda_1^2 + \lambda_2\sigma_1\right) u(\eta_1) + \lambda_2 \left(\lambda_1 + \sigma_2\right) u(\eta_2) + \lambda_2\sigma_3 u(\eta_3).$$

Comparing coefficients against those in Equation (5.9), we find

$$\begin{aligned} \lambda_0\lambda_1 &= \tau_0 \\ \lambda_0 + \lambda_1^2 + \lambda_2\sigma_1 &= \tau_1 \\ \lambda_2\left(\lambda_1 + \sigma_2\right) &= \tau_2 \\ \lambda_2\sigma_3 &= \tau_3 \end{aligned}$$

so that $\lambda_2, \tau_0, \tau_3$ are determined by knowledge of λ_0, λ_1, τ_1 and τ_2. Now all we need are the eigenvalues needed in Proposition 4.7. But we know $\eta_0 = k$, the valency of Γ, we are given η_1 and η_2 by hypothesis and we may then solve for η_3 by looking at the $i = 1$ entry on both sides of (5.9):

$$\tau_0 + \tau_1\frac{\eta_1 - \alpha_0}{k - \alpha_0} + \tau_2\frac{\eta_2 - \alpha_0}{k - \alpha_0} + \tau_3\frac{\eta_3 - \alpha_0}{k - \alpha_0} = \left(\frac{\eta_1 - \alpha_0}{k - \alpha_0}\right)^3$$

where we have used the evaluation (3.7) $u_1(\theta) = (\theta - \alpha_0)/(k - \alpha_0)$. Now the result follows from Proposition 4.7. $\square$

CONJECTURE 5.6. *Every completely regular code in a Q-polynomial distance-regular graph with sufficiently large covering radius is a Leonard completely regular code.*

In [**19**], Terwilliger gave a parametrization of all Leonard pairs along the lines of the Askey scheme. This is closely related to the classification of parameter sets for Q-polynomial distance-regular graphs found in [**1**, Theorem III.5.1]. This parametrization now gives us new terminology for completely regular codes.

DEFINITION 5.7. We say a Leonard code is of type *Krawtchouk* if the corresponding Leonard pair is of type *Krawtchouk* as defined in Terwilliger [**19**]. In a similar fashion, we define Leonard codes of type *Hahn*, *dual Hahn*, *Racah* and so on.
Sometimes we also say that a Leonard code is of *class* (I), (IA), (IB), (II), (IIA), (IIB), (IIC), (IID) and (III) if the corresponding Leonard pair is of class (I), (IA), (IB), (II), (IIA), (IIB), (IIC), (IID) and (III), respectively, where we use the notation of Bannai and Ito [**1**].

It is a natural problem to choose one of these families and to classify all Leonard codes of that type. It is interesting to note that a Leonard code of a given type may appear within a classical distance-regular graph of some other type. For example, the n-cube is obviously a Q-polynomial distance-regular graph of Krawtchouk type, and it contains the binary repetition code, which is not of Krawtchouk type. Below, in Example 5.8, we describe additive binary completely regular codes found by Rifà and Zinoviev [**14**] which are of dual Hahn type. In the next example, Example 5.9, we describe linear q-ary completely regular codes found in Rifà and Zinoviev [**15**], whose quotient graph is a bilinear forms graph. This last example shows that even the class of a Leonard code may be different from the class of the classical distance-regular graph, it appears in.

EXAMPLE 5.8. In any $\binom{m}{2}$-cube for integer $m \geq 3$, there exist Leonard completely regular codes which are not of Krawtchouk type. Following [**14**], for natural numbers $m \geq 3$ and $2 \leq l < m$, define E_l^m as the set of all binary vectors of length m and weight l. Denote by $H^{(m,l)}$ the binary matrix of size $m \times \binom{m}{l}$, whose columns are exactly all vectors from E_l^m. Rifà and Zinoviev consider the binary linear code $C^{(m,l)}$ whose parity check matrix is the matrix $H^{(m,l)}$; they show that the code $C^{(m,2)}$ is completely regular and its coset graph is the halved m-cube. As the halved m-cube is Q-polynomial, it follows that $C^{(m,2)}$ is Leonard, but it is of dual Hahn, not Krawtchouk, type.

Example 5.9. Let m be a positive integer and q a prime power. Let H_m be the parity check matrix of the Hamming code over the field with q elements, GF(q), with length $n = \frac{q^m-1}{q-1}$. Let $\ell \geq 2$ and let $r = q^\ell$. Let $C'(m,\ell)$ be the r-ary code whose parity check matrix is H_m. In [**15**], it is shown that $C'(m,\ell)$ is completely regular with covering radius $\min(m,\ell)$ and the quotient graph is a bilinear forms graph. Clearly, the Hamming graphs are of Krawtchouk type, whereas $C'(m,\ell)$ is of q-Krawtchouk type.

6. Harmonic completely regular codes

In a companion paper [**10**], we explore a well-structured class of Leonard completely regular codes in the Hamming graphs. These *arithmetic* completely regular codes are defined as those whose eigenvalues are in arithmetic progression: $\mathrm{Spec}\,(C) = \{k, k-t, k-2t, \ldots\}$. These codes have a rich structure and are intimately tied to Hamming quotients of Hamming graphs. In [**10**], we study products of completely regular codes and completely classify the possible quotients of a Hamming graph that can arise from the coset partition of a linear arithmetic completely regular code. For families of distance-regular graphs other than the Hamming graphs, we need to look at a slightly weaker definition to probe the same sort of rich structure.

We next introduce the class of *harmonic* completely regular codes and we will see that this class lies strictly between the arithmetic completely regular codes and the Leonard completely regular codes.

Definition 6.1. Let Γ be a Q-polynomial distance-regular graph with respect to the ordering $\theta_0, \theta_1, \ldots, \theta_D$ of its eigenvalues and C be a completely regular code of Γ. We call the code C *harmonic* if $\mathrm{Spec}\,(C) = \{\theta_{ti} \mid i = 0, \ldots, \rho\}$ for some positive integer t.

Let Γ be a Q-polynomial with respect to the ordering $\{\theta_0, \theta_1, \ldots, \theta_D\}$ of its eigenvalues and let $C \subseteq V\Gamma$ be a code. Then *strength* of C, $t(C)$ is defined as the $\min\{i \geq 1 \mid \theta_i \in \mathrm{Spec}^*(C)\} - 1$.

Example 6.2. The following are examples of harmonic completely regular codes:
(1) the repetition code in a hypercube;
(2) cartesian products of a completely regular code of a Hamming graph $C \times \cdots \times C$ where C is covering radius 1;
(3) in the Grassmann Graph $J_q(n,t)$, whose vertices are all t-dimensional subspaces of a some n-dimensional vector space V over $GF(q)$, we find the following two families:

- C consists of all t-dimensional subspaces of a given $(n-s)$-dimensional subspace of V, where $0 < s < n-t$;
- C consists of all t-dimensional subspaces of V containing a fixed s-dimensional subspace U of V, where $0 < s \leq t < n$.

(We note that the Johnson graph $J(n,t)$ contains examples analogous to these.)
(4) any completely regular code of strength 0 in a Q-polynomial distance-regular graph.

LEMMA 6.3. *Let Γ be a Q-polynomial distance-regular graph with respect to the ordering $\theta_0, \theta_1, \ldots, \theta_D$ of its eigenvalues. Then any harmonic completely regular code is a Leonard completely regular code.*

PROOF. Assume C is a completely regular code in Γ with $\mathrm{Spec}\,(C) = \{\theta_{ti} \mid i = 0, \ldots, \rho\}$ for some positive integer t. Since Γ is Q-polynomial, there exist numbers $\omega_{h,j}$ such that $E_t\mathbf{x} \circ E_{jt}\mathbf{x} = \sum_{h=0}^{\rho} \omega_{h,j} E_{ht}\mathbf{x}$ and the following holds:

$$\omega_{h,j} \begin{cases} = 0 & \text{if } |ht - jt| > t \\ \neq 0 & \text{if } |ht - jt| \leq t \end{cases}.$$

So,

$$\omega_{h,j} \begin{cases} = 0 & \text{if } |h - j| > 1 \\ \neq 0 & \text{if } |h - j| \leq 1 \end{cases}.$$

Hence the matrix representing $\mathcal{A}^*(\theta_t)$ is irreducible tridiagonal with respect to $\mathcal{B}$. So, by Theorem 5.4, C is Leonard. □

Finally, we remark that the codes given in Example 5.8 are Leonard but not harmonic.

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