Completely regular codes – a viewpoint and some problems

W. J. Martin

Department of Mathematical Sciences and Department of Computer Science
Worcester Polytechnic Institute, Worcester, Massachusetts
martin@wpi.edu

Abstract. Since their introduction in 1973, completely regular codes have been of interest to coding theorists and graph theorists alike. These highly regular substructures were defined as a generalization of perfect and uniformly packed error-correcting codes but also include many codes having very small minimum distance which are fundamental to the study of distance-regular graphs. While interest in these codes among coding theorists seems to be on the decline, there is reason to believe that the importance of completely regular codes to the theory of distance-regular graphs has yet to be fully realized. This paper is an attempt to tell this story and explain these trends. Some open problems are discussed and a bibliography of recent literature is included.

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1 Introduction: Completely regular codes in coding theory

In 1973, in the context of the search for extremal error-correcting codes, culminating in the classification by van Lint and Tietäväinen of perfect codes over finite fields, together with the discovery of the last few sporadic finite simple groups, Philippe Delsarte introduced a class of codes which enjoy combinatorial (and often algebraic) symmetry akin to that observed in perfect codes. At the outset, completely regular codes lived a sort of dual life. By "day", this was to be a class of codes which would be useful for practical error correction applications. Delsarte observed that all perfect codes are completely regular and, after a bit of terminological adjustment, it was agreed that nearly perfect and uniformly packed codes are succinctly described as those completely regular codes whose covering radius exceeds their packing radius by one. (This family includes the Preparata codes, for example.) Perhaps some coding theorists of the day felt that there were more extremal error-correcting codes to be found in the guise of completely regular codes. It seemed natural to expect that a very large code with very large minimum distance would have a great deal of symmetry. Recent work on completely transitive codes can be found in [22, 1].

Yet, from the very beginning, completely regular codes lived a more esoteric "night" life. Delsarte gave the definition not only for codes in Hamming graphs, but for "codes" in arbitrary distance-regular graphs. (And, as we shall see, he initiated the study of a wider class of codes — those which we will call "simple codes" — in association schemes which are not necessarily P-polynomial.) It was Delsarte who posed the question of existence of non-trivial perfect codes in the Johnson graphs.

Independently, Biggs and co-workers Smith and Hammond were exploring perfect codes in distance-regular graphs, extending the definitions and machinery (such as Lloyd's theorem) beyond the Hamming graphs, which is essentially the only case of practical interest to coding theorists. For example, their work led to a classification of all perfect codes in cubic distance-regular graphs. See [3] for a full bibliography up to 1989.

First, the story from the coding theory viewpoint. After the classification of perfect codes was complete, van Tilborg and others went to work on the next most interesting case: uniformly packed codes. In his 1976 thesis, van Tilborg proved that every non-trivial uniformly packed code has packing radius $e \leq 3$. He also determined all uniformly packed codes with $e = 3$ and a subclass of those with $e = 1, 2$. The case of binary linear uniformly packed codes with $e = 2$ was settled by Calderbank and Goethals in 1985 with one final case resolved by Calderbank in 1986. The classification is described in detail in [3, Sec. 11.1]. Examples include the Preparata codes, a family of 2-error-correcting binary BCH codes, many single-error-correcting codes, binary repetition codes in $2m$-cubes, and four codes from the Golay family.

Other completely regular codes in the Hamming graphs include the Kasami codes, the extended Preparata codes, a few more codes from the Golay family, a few Hadamard codes of length 12 or less, and many more if one allows minimum distance three or less. Extending work of Rifà and Huguet, Brouwer, et al. [3] prove that any translation distance-regular graph of diameter at least three defined on an elementary abelian group is the coset graph of some completely regular code in some Hamming graph. See [35, 36, 38, 39] for papers containing results of this sort.

But from a coding theorist's viewpoint, this harvest was disappointing. Apart from the perfect codes and the codes listed above, there have been no completely regular codes found with promising error-correcting capabilities. In [2], we read that "it has been conjectured for a long time that if C is a completely regular code and $|C| > 2$, then $e \leq 3$." In [34], Neumaier conjectured that the only completely regular codes with minimum distance $d(C) \geq 8$ are the binary repetition codes and the extended binary Golay code. While these conjectures are of interest in the theory of distance-regular graphs, they signal the end of the story for algebraic coding theorists.

Before we change direction entirely, and before we stop to give the relevant definitions, we mention a few more problems related to completely regular codes in the Hamming graphs. The general idea is to restate fundamental problems regarding distance-regular graphs in the narrower context of coset graphs:

- (Brouwer, et al. [3, p357]) Is it true that a distance-regular coset graph (which is necessarily the coset graph of some completely regular code) with classical parameters is one of the known graphs?
- Is it true that a distance-regular coset graph of sufficiently large diameter is Q-polynomial?
- Does Neumaier's conjecture hold for additive completely regular codes? (I.e., for distance-regular coset graphs, is there is an upper bound on i such that $c_j = j$ and $a_j = a_1 j$ for all $j \leq i$ with the exception of Hamming (and Doob) graphs themselves?

2 Definitions

Let Γ denote a distance-regular graph of diameter d on vertex set X with adjacency matrix A, distance matrices $A_0, \ldots, A_d$ and Bose-Mesner algebra $\mathbb{A}$ acting on $V = \mathbb{R}^X$. For $x \in X$ and $0 \le i \le d$, write

$$\Gamma_i(x) = \{y \in X : \partial(x,y) = i\}$$

where $\partial(x,y)$ is the length of a shortest path from x to y in the undirected graph Γ. For $C \subseteq X$, we consider the *minimum distance* of C (when $|C| > 2$)

$$\delta = d(C) = \min\{\partial(x,y) : x, y \in C,\ x \ne y\}$$

and the *covering radius* of C

$$\rho = \rho(C) = \max\{\partial(x,C) : x \in X\}$$

where the distance from vertex x to the set C is $\partial(x,C) = \min\{\partial(x,c) : c \in C\}$. The *degree* of C is the number of non-zero distances that occur between pairs c, c' in C.

The space V naturally decomposes into pairwise orthogonal maximal eigenspaces V_0, …, V_d for the matrix A: for $\mathbf{v} \in V_j$, $A\mathbf{v} = \theta_j \mathbf{v}$ where, by convention, we will assume that $\theta_0 > \theta_1 > \cdots > \theta_d$ are the distinct eigenvalues of A in decreasing order. Let E_j denote the matrix representing orthogonal projection from V onto V_j. We will be interested in a certain A-module associated to the code C. Let $\mathbf{x} \in V$ denote the characteristic vector of C: the entry in position c is one if $c \in C$ and zero otherwise. The *outer distribution module* of C is the subspace

$$\mathbb{A}\mathbf{x} = \{M\mathbf{x} : M \in \mathbb{A}\}$$

of V. This is seen to be the column space of the *outer distribution matrix* D whose rows are indexed by X and whose columns are labelled $0, 1, \ldots, d$. The entry in row u, column i is the number of codewords at distance i from u: $D_{u,i} = |\Gamma_i(u) \cap C|$. Hence the i^{th} column of D is $A_i\mathbf{x}$ and since the A_i span $\mathbb{A}$, we have $\mathbb{A}\mathbf{x} = \mathsf{colsp} D$. The integer $s^* = \mathsf{rank}(D) - 1$ is called the *dual degree* of C. It is an easy exercise to show that $\rho \le s^*$ for any code C. The case of equality is sometimes interesting.

A code $C \subseteq X$ is *completely regular* if there exist constants r_{ij} ($0 \le i \le \rho$, $0 \le j \le d$) such that, whenever $\partial(x,C) = i$, the entries in row x of D are $r_{i0}, r_{i1}, \ldots, r_{in}$; that is, the (u,j)-entry of D depends only on j and the distance from u to C and not on u itself. Delsarte introduced this concept in 1973, showing that the class of completely regular codes contains all perfect codes and all uniformly packed codes (in any distance-regular graph).

Any code C determines a natural partition of the vertex set X according to distance from C. Define, for $0 \le i \le \rho$,

$$C_i = \{u \in X : \partial(u,C) = i\}.$$

Then $\pi = \{C_0, C_1, \ldots, C_\rho\}$ is the *distance partition* of X with respect to C. Such a partition is called *equitable* [18, p75] if the number of vertices in C_j adjacent to a vertex in C_i depends only on i and j and not on the choice of vertex. By the triangle inequality,a vertex in C_i has all its neighbors in $C_{i-1} \cup C_i \cup C_{i+1}$; so for this particular type of partition, most of these values are zero. (It will be convenient to define $C_{-1} = C_{\rho+1} = \emptyset$.)

Theorem 1 (Cf. Neumaier [34]). *Let Γ be a distance-regular graph with vertex set X and let C be a non-empty subset of Γ. With notation as above, the following are equivalent:*

(i) *C is a completely regular code;*
(ii) *the distance partition π of X with respect to C is equitable;*
(iii) *the outer distribution module $\mathbb{A}\mathbf{x}$ is closed under entrywise multiplication and satisfies $\rho = s^*$.*

Proof ((i)$\Rightarrow$ (iii)). Let $\mathbf{x}_i$ denote the characteristic vector of the cell C_i of the distance partition. If C is completely regular, then each column of D is a linear combination of $\mathbf{x}_0, \ldots, \mathbf{x}_\rho$. So the span of these $\rho + 1$ vectors contains $\mathbb{A}\mathbf{x}$. But $\rho + 1 \leq s^* + 1 = \dim \mathbb{A}\mathbf{x}$. So the module admits a basis of $\rho + 1$ pairwise orthogonal 01-vectors. Now (iii) follows.

[(iii) $\Rightarrow$ (i)] If (iii) holds, then the outer distribution module admits a basis of $\rho + 1$ pairwise orthogonal 01-vectors. By induction, starting with the basis $\{\mathbf{x}, A\mathbf{x}, \ldots, A^\rho \mathbf{x}\}$, one quickly sees that these must be $\mathbf{x}_0, \ldots, \mathbf{x}_\rho$. Thus every column of D is a linear combination of $\mathbf{x}_0, \ldots, \mathbf{x}_\rho$ and (i) holds.

[(iii) $\Rightarrow$ (ii)] Since (iii) holds, we know that $\mathbf{x}_0, \ldots, \mathbf{x}_\rho$ is a basis for $\mathbb{A}\mathbf{x}$. So, for each i, $A\mathbf{x}_i$ can be expressed as a linear combination of $\mathbf{x}_0, \ldots, \mathbf{x}_\rho$. Interpreting this combinatorially, we find that partition π is equitable.

[(ii) $\Rightarrow$ (iii)] If π is equitable, then $A\mathbf{x}_i$ is a linear combination of $\mathbf{x}_{i-1}$, $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ (with $x_{-1} = \mathbf{x}_{\rho+1} = \mathbf{0}$). So the space W spanned by the $\mathbf{x}_i$ is A-invariant. Since $\mathbb{A}$ is generated by A and $\mathbf{x} \in \mathbb{A}\mathbf{x}$, W is equal to $\mathbb{A}\mathbf{x}$. □

In light of this theorem, we use the three characterizations interchangeably. Each viewpoint aids in the proof of some basic result, as we now show.

Proposition 1 (Neumaier [34]). *Let Γ be a distance-regular graph with valency k and let C be a completely regular code in Γ with covering radius ρ. Then C_ρ is also completely regular.*

Proof. Immediate from (ii). □

Theorem 2 (Delsarte [12]). *For any code C having dual degree s^* and covering radius ρ in a distance-regular graph Γ, we have $\rho \leq s^*$ and $d(C) \leq 2s^* + 1$. If $d(C) \geq 2s^* - 1$, then C is completely regular.*

Proof. Let $u_0, u_1, \ldots, u_\rho$ be vertices of Γ with $u_i \in C_i$. Then the submatrix M of D determined by restricting to these $\rho + 1$ rows (in this order) has $M_{ii} > 0$ and $M_{ij} = 0$ for $j < i$ ($0 \leq i \leq \rho$). So $s^* + 1 = \mathsf{rank} D \geq \mathsf{rank} M = \rho + 1$. Clearly, $d(C) \leq 2\rho + 1$, so $d(C) \leq 2s^* + 1$. Now if $d(C) = 2s^* + 1$, then columns $0, 1, \ldots, \rho$ of D form a basis for $\mathbb{A}\mathbf{x}$. So the entry in row u, column i of D depends only on the entries in row u, columns $0, \ldots, \rho$, which in turn depend only on $d(u, C)$. Thus C is completely regular. If $d(C) = 2s^* - 1$ or $2s^*$, then columns $0, \ldots, s^* - 1$ are linearly independent and appending the all-ones vector (the sum of all columns of D) to this set again yields a basis consisting of vectors whose u-entries depend only on $d(u, C)$. □

Question: Is there any additional hypothesis which implies complete regularity when $d(C) = 2s^* - 2$?

Following [5], we define the *width* of a subset C in a distance-regular graph Γ to be the maximum distance between any two vertices of C: $w = \max\{\partial(c, c') | c, c' \in C\}$.

Theorem 3 ([5]). *For any code C having width w and dual degree s^* in a distance-regular graph Γ of diameter d, we have $w + s^* \geq d$. If $w + s^* = d$, then C is a completely regular code.* □

The case where $w + s^* = d$ includes many fundamental substructures in the classical distance-regular graphs [13]. Further information about such substructures helps us to understand the classical familes. If Γ is also Q-polynomial, one may also define the *dual width* of C as $w^* = \max\{j | E_j x \neq 0\}$ where x is the characteristic vector of C and the primitive idempotents are in a Q-polynomial ordering. It is known that, for any code C in a Q-polynomial distance-regular graph of diameter d, we have $w + w^* \geq d$. We expect a nice classification of those codes satisfying $w + w^* = d$ in the classical families: Hamming, Johnson and their q-analogues.

3 Feasibility conditions

Let C be a completely regular code with equitable distance partition

$$\pi = \{C_0, C_1, \ldots, C_\rho\}.$$

Then, with $C_{-1} = C_{\rho+1} = \emptyset$, there exist integers $\alpha_i, \beta_i, \gamma_i$ $(0 \leq i \leq \rho)$ such that each vertex in C_i has α_i neighbors in C_i, β_i neighbors in C_{i+1}, and γ_i neighbors in C_{i-1}. The *intersection array* of C is the ordered pair of sequences

$$\iota(C) = \{\beta_0, \beta_1, \ldots, \beta_{\rho-1}; \gamma_1, \gamma_2, \ldots, \gamma_\rho\}.$$

Typically the graph Γ (or, as in the case of codes in the Hamming graphs, a family of graphs Γ) is understood from the context.

Little progress has been made toward establishing feasibility conditions for an intersection array $\{\beta_0, \ldots, \beta_{\rho-1}; \gamma_1, \ldots, \gamma_\rho\}$. There are obvious conditions based on the fact that the $|C_i|$ must be integers summing to $|X|$. In some cases, the local structure of the graph Γ imposes restrictions on adjacent β_i and γ_j. The strongest feasibility condition, by far, is Lloyd's Theorem, which is presented in the next section.

A natural condition to expect is monotonicity:

$$\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_\rho, \qquad \beta_0 \geq \beta_1 \geq \cdots \geq \beta_{\rho-1}. \tag{1}$$

The following simple example shows that this property does not necessarily hold for completely regular codes in arbitrary distance-regular graphs:
Example: Let Γ be the dodecahedron and let C be the vertex set of any pentagon in Γ. Then C is a completely regular code with intersection array $\{1, 2, 1; 1, 2, 1\}$.

This example – along with an infinite family of others violating the monotonicity condition – was given in [26] along with the perhaps surprising result that the monotonicity conditions *do* hold for completely regular codes in many families of classical distance-regular graphs. Koolen proves that, if the graph Γ admits a non-identity automorphism α with $\partial(y, \alpha y) \leq 1$ for all vertices y, then (1) holds for any completely regular code in Γ. For instance, it follows that the monotonicity condition holds for all completely regular codes in a Hamming or Johnson graph. In [27], a stronger monotonicity condition is given in the case of the Hamming graphs.

4 The spectrum of a code

The next result is the natural generalization of Lloyd's theorem to completely regular codes.

Theorem 4 (Neumaier [34]). *For any completely regular code C with covering radius ρ in a distance-regular graph Γ, the tridiagonal matrix*

$$B = \begin{bmatrix} \alpha_0 & \beta_0 & & & \\ \gamma_1 & \alpha_1 & \beta_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \gamma_{\rho-1} & \alpha_{\rho-1} & \beta_{\rho-1} \\ & & & \gamma_\rho & \alpha_\rho \end{bmatrix}$$

has $\rho+1$ distinct eigenvalues and each of these is an eigenvalue of Γ.

Proof. Using Theorem 1(iii), matrix B represents the action of the adjacency matrix A on the outer distribution module $\mathbb{A}\mathbf{x}$ with respect to the basis $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\rho$. □

The eigenvalues of B, usually written in decreasing order, form what we call the *spectrum* of the code: $\mathsf{spec}(C) = \{\theta_0, \theta_{j_1}, \ldots, \theta_{j_\rho}\}$. The *dual degree set* of C is the set $S^*(C) = \{j_1, \ldots, j_\rho\}$ of non-zero subscripts of those eigenvalues of Γ appearing in $\mathsf{spec}(C)$.

We note that the intersection array $\iota(C)$ and the graph Γ determine B which, in turn determines the entries r_{ij} of the (reduced) outer distribution matrix. First, use the recurrence

$$BB_i = b_{i-1}B_{i-1} + a_iB_i + c_{i+1}B_{i+1}$$

where $B_{-1} = 0$, $B_0 = I$, $B_1 = B$ and the a_i, b_i and c_i are the intersection numbers of graph Γ to determine $B_2, \ldots, B_d$. (The entry in row h, column j of B_i is the number of vertices in C_j at distance i from any vertex in C_h.) Then $[r_{0,i}, \ldots, r_{\rho,i}]^\perp$ is the zero-th column of B_i. We can go in the reverse direction as well: the parameters r_{ij} determine the matrix B.
Example: The Odd graph O_k has valency k, diameter $k-1$ and no 4-cycles. If C is a perfect 2-code in O_k, then the above theorem implies that the matrix

$$B = \begin{bmatrix} 0 & k & 0 \\ 1 & 0 & k-1 \\ 0 & 1 & k-1 \end{bmatrix}$$

has integer eigenvalues. It follows that $k = s^2+s+1$ for some odd integer s and the $(k-1)$-sets comprising C form a block design with strength $t = s^2-1$ on $v = 2s^2+2s+1$ points consisting of $|C| = \binom{2s^2+2s+1}{s^2+s}/(s^2+1)(s^2+2s+2)$ blocks. No examples are known with $s > 1$ yet no proof of non-existence has yet been produced. This situation is explored further in [23]. We complete our discussion of this example with the following

Theorem 5 (Martin [28]). *If C is a perfect code in the Odd graph O_k, then C is also a completely regular code in the Johnson graph $J(2k-1, k-1)$ defined on the same vertex set.* □

As in the above example, one tool that comes to our aid in the study of completely regular codes is combinatorial design theory. In a Q-polynomial scheme with Q-polynomial ordering $0, 1, \ldots, d$, a subset C of X is a *Delsarte t-design* if its characteristic vector $\mathbf{x}$ satisfies $E_j\mathbf{x} = \mathbf{0}$ for $j = 1, \ldots, t$. Celebrated results of Delsarte give combinatorial characterizations of Delsarte t-designs in the Hamming scheme (here, they are orthogonal arrays of strength t), the Johnson scheme (t-(v, k, λ) block designs), and their q-analogues. Ironically, it seems that the lower strength case is easier for us to handle.

Lemma 1. *Let C be a completely regular code in distance-regular graph Γ with dual degree set $S^*(C)$. Then, for $j > 0$, we have $E_j\mathbf{x} \neq \mathbf{0}$ if and only if $j \in S^*(C)$.*

Proof. Let P be the $|X| \times (\rho + 1)$ matrix whose i^{th} column is $\mathbf{x}_i$, the characteristic vector of C_i. Then $AP = PB$ and if $\mathbf{u}$ is any right eigenvector for B belonging to eigenvalue θ, then $\mathbf{v} := P\mathbf{u}$ is a right eigenvector for A with the same eigenvalue. Since we can always take $\mathbf{u}$ to have first entry 1, $\mathbf{x}$ is not orthogonal to any of these $\rho + 1$ eigenspaces. Since $\{E_j\mathbf{x} : 0 \leq j \leq d\}$ spans $\mathbb{A}\mathbf{x}$, a space of dimension $\rho + 1$, we are done. □

In [27], it is shown that if Γ is Q-polynomial with ordering $0, 1, \ldots, d$ and the elements of the dual degree set of C are $j_1 < j_2 < \ldots < j_\rho$, then $j_h - j_{h-1} < j_1$ for each $h = 2, \ldots, \rho$. This rules out many parameter sets for completely regular codes in the Hamming graphs.

I should mention that Fiol and Garriga have some interesting results which characterize completely regular codes as extremal subsets with respect to spectral phenomena in [16, 15]. Their work makes use of orthogonal polynomials.

5 Codes with small strength

Meyerowitz [32, 33] classified the completely regular codes of strength zero in the Hamming graphs and the Johnson graphs. (He also found all completely regular codes in the complete multipartite graphs.)

Theorem 6 (Meyerowitz). *Let C be a completely regular code in the Hamming graph $H(n, q)$ with $E_1\mathbf{x} \neq 0$. Then C is isomorphic to a code of the form*

$$C' = \{y = (y_1, \ldots, y_n) \mid y_1, y_2, \ldots, y_\rho < \gamma_1\}.$$

Proof. Let $\mathbf{w}_{i,a}$ denote the characteristic vector of the completely regular code consisting of all q-ary n-tuples whose i^{th} coordinate is equal to a. Then

$$\{\mathbf{w}_{i,a} \mid 1 \leq i \leq n,\ a = 0, 1, \ldots, q - 1\}$$

is a set of 01-vectors which spans $V_0 \oplus V_1$. (One easily checks that each of these codes has dual degree set $\{1\}$ and that each character in V_1 can be expressed as a linear combination of the $\mathbf{w}_{i,a}$.

Next, Meyerowitz shows that a vector $\mathbf{u}$ in V_1 can be uniquely expressed as

$$\mathbf{u} = \sum_{i=1}^{n} \sum_{a=0}^{q-1} \tau_{i,a} \mathbf{w}_{i,a}$$

subject to the restriction that $\sum_a \tau_{i,a} = 0$ for each i. Let $\mathbf{u} = E_1\mathbf{x}$ and consider this expansion for $\mathbf{u}$. The entry indexed by a q-ary n-tuple y is

$$\mathbf{u}_y = \sum_{i=1}^{n} \tau_{i,y_i}.$$

Now we also have $\mathbf{u} = \omega_0\mathbf{x}_0 + \omega_1\mathbf{x}_1 + \cdots$ since $E_1\mathbf{x} \in \mathbb{A}\mathbf{x}$. The coefficients $\omega_0, \omega_1, \ldots$ are the entries of some right eigenvector of B belonging to eigenvalue θ_1. Let us scale $\mathbf{u}$ so that $\omega_0 = 1$. Then

$$\alpha_0\omega_0 + \beta_0\omega_1 = \theta_1\omega_0$$

giving $\omega_1 = (\theta_1 - \alpha_0)/(k - \alpha_0) < 1$. Next we see that an n-tuple y of Hamming weight one — with $y_j = a \neq 0$, say — satisfies

$$\mathbf{u}_y = \sum_{i \neq j} \tau_{i,0} + \tau_{j,a} = 1 + (\tau_{j,a} - \tau_{j,0}).$$

So if $y \in C$, we have $\tau_{j,a} = \tau_{j,0}$ and, otherwise, $y \in C_1$ and $\tau_{j,a} = \tau_{j,0} - (1 - \omega_1) < \tau_{j,0}$.

Since $y \in C$ requires $\mathbf{u}_y = 1$, we find that $y \in C$ only if $\tau_{i,y_i} = \tau_{i,0}$ for all i. It is easy to check that this condition is also sufficient: let $0 = z^0, z^1, \ldots, z^\ell = y$ be a geodetic path from 0 to y; each $\tau_{i,z_i^j} = \tau_{i,0}$ so no z^j lies in C_1; thus all lie in C_0.

Next, let y be a vertex of Hamming weight one lying in C_1. Then y has γ_1 neighbors in C and these must all differ from y in the same coordinate. It follows that, for each i, either all $\tau_{i,a} = \tau_{i,0}$ or exactly γ_1 values of a satisfy this condition. □

For the Johnson graph $J(v, k)$, Meyerowitz proved that each completely regular codes having strength zero can be described as the collection of all k-sets incident with some fixed subset T of the ground set. The analogous results for the Grassman and bilinear forms graphs have yet to be found. I should give a word of warning: in the Grassman graph $G_q(4, 2)$, there is at least one extra example. Consider a projective point P and a hyperplane H skew to it. Then the code C consisting of all lines incident to either one is completely regular of strength zero. So at least some adjustment is required to move from the result for $J(v, k)$ to the corresponding classification for $G_q(v, k)$.

Following this idea of Meyerowitz — that of describing eigenspaces via 01-spanning sets — Martin [29] found all completely regular codes in $J(v, k)$ having strength one and minimum distance greater than one. The minimum distance one case remains unresolved. For the Hamming graphs, Martin [31] proved that any completely regular code with strength t has minimum distance at most $2t + 1$, with the exception of binary repetition codes. Something similar should hold for the Johnson graphs and perhaps other classical families. It seems that the locally sparse graphs are the only ones that admit completely regular codes having large minimum distance.

6 Completely regular partitions

Algebraically, the concept of a completely regular code is dual to that of a (cometric) induced association scheme. This duality is simplest in the case of the Hamming graphs: a linear code

C is completely regular if and only if the Hamming relations, restricted to the dual code $C^{\perp}$, form a cometric association scheme on $C^{\perp}$. Moreover, the distance-regular coset graph obtained from C and this induced scheme form a dual pair of association schemes.

In an antipodal distance-regular graph Γ, any non-empty subset of an antipodal class is completely regular. Moreover, the partition of Γ into antipodal classes is a "completely regular" partition [3, p351] with the property that the folded graph $\bar{\Gamma}$ can be viewed as having these completely regular codes as its vertices, two being joined by an edge precisely when there exists an edge of Γ from a vertex in one to a vertex in the other. (See Sec. 11.1 of [3] for full details.) This seems to be just the tip of an iceberg. As stated in the introduction, there are other ways to form a distance-regular quotient graph than simply collapsing antipodal classes. Yet, aside from coset graphs — quotients of the Hamming graphs — I know of few examples.

As stated above, completely regular partitions of the Hamming graphs are intrinsic to the study of translation distance-regular graphs. Aside from the folded Johnson graphs and some yielding trivial quotients, I know of no completely regular partitions of the Johnson graphs and have some partial results in [28]. When such a quotient exists, its completely regular codes all lift back to the original graph.

Theorem 7 ([28]). *Let Γ be a distance-regular graph with vertex set X and let $\sigma = \{C_1, \ldots, C_m\}$ be a completely regular partition of X giving rise to the distance-regular quotient graph $\bar{\Gamma}$ with vertex set σ. Let $\psi : X \to \sigma$ be the natural map sending each vertex of Γ to the fibre containing it.*

- *If S is any completely regular code in $\bar{\Gamma}$, then $\psi^{-1}(S)$ is a completely regular code in Γ;*
- *if S is a completely regular code in Γ which is expressed as a union of fibres, then $\psi(S)$ is a completely regular code in $\bar{\Gamma}$.*

Proof. Straightforward. □

Consider the three Moore graphs of diameter two and a hypothetical fourth such graph of valency fifty-seven. Each known graph appears in each larger known graph as a completely regular code and in fact we have completely regular partitions: the quotient of the Petersen graph over two disjoint pentagons yields K_2; collapsing the Hoffman-Singleton graph over a completely regular partition into five Petersen graphs yields K_5 as a quotient; the quotient of the Hoffman-Singleton graph over a completely regular partition into ten pentagons is the complete bipartite graph $K_{5,5}$.

If a Moore graph of valency 57 exists, the quotient over a completely regular partition into pentagons would yield an $SRG(650, 55; 0, 5)$ — this parameter set is currently open. Likewise, if the putative graph of valency 57 admits a completely regular partition into Petersen subgraphs, the quotient would be an $SRG(325, 54; 3, 10)$ — again an open case. It is easy to see that a Moore graph of valency 57 could not admit a completely regular partition into Hoffman-Singleton graphs as the quotient would be a strongly regular graph on 65 vertices with valency $k = 50$ and $a_2 = 0$.

Having made these lofty statements, I must admit that, while our hypothetical Moore graph is guaranteed to contain plenty of pentagons, I cannot even prove that it must contain a single Petersen graph. This was posed as an open problem by Godsil in [19].

7 Small covering radius

Canogar [8] and Fon-Der-Flaass [17] independently provided constructions for completely regular codes having covering radius one in the n-cube. They each prove that the obvious necessary conditions for the feasibility of the array $\{\beta_0;\gamma_1\}$ are asymptotically sufficient, in the sense that, for sufficiently large a, the array $\begin{bmatrix} a & \beta_0 \\ \gamma_1 & a+\beta_0-\gamma_1 \end{bmatrix}$ is the intersection matrix of some completely regular code. Fon-Der-Flaass further points out that, for n large enough, such a code must be the product of a trivial code and a code with the same intersection array in the $(n-1)$-cube. It seems feasible to extend this analysis to the Hamming graphs $H(n,q)$ for $q>2$. Note that no attempt is made in these papers to classify codes up to isomorphism; even the case of perfect 1-codes is unwieldy.

Classifying completely regular codes with $\rho=1$ in the Johnson graphs includes a classification of all Steiner triple systems and all Steiner quadruple systems [30]. The spectrum of parameters for such designs is however known. The following problem further suggests that the case of the Johnson graphs will be much harder to handle than the Hamming graphs. What has become to be known as "Delsarte's conjecture" is the claim that there are no perfect e-codes in the Johnson graphs with $e\ge 1$ and $|C|>2$. Roos proved the necessary condition $k-1\ge ev/(2e+1)$. (We may assume, without loss, that $k\le v/2$.)

For $e=1$, we find that there exist integers r and s for which $k=rs+1$, $v=2rs+r-s+1$ and the perfect 1-code is a block design of strength $t=(r-1)s$. The derived design, consisting of all blocks of C which contain any specified symbol is also a completely regular code. This code has intersection array $\{(k-1)(v-k), v-k-1; 1, (k-1)(v-k-1)\}$ and Lloyd's Theorem holds vacuously since the eigenvalues of the 3×3 tridiagonal matrix B for this code are the valency, $(k-1)(v-k)$, and s and $-r$. More generally, we have

Theorem 8 (Martin [28]). *If C is a non-trivial perfect code in $J(v,k)$, then the derived design*

$$C' = \{x\setminus\{1\} \;:\; 1\in x\in C\}$$

forms a completely regular code in the Johnson graph $J(v-1,k-1)$. □

Thus Lloyd's Theorem for the derived design provides an additional number-theoretic condition on the existence of the perfect code. For a more interesting example, suppose C is a perfect 2-code in $J(v,k)$. Then Lloyd's Theorem for C implies that W given by

$$W^2 = (v-2)(v-10)+4k(v-k)$$

is an integer. (Note that trivial perfect 2-codes exist in $J(v,2)$ and $J(10,5)$.) The intersection numbers for the derived design in this case are

$$\gamma_i = \begin{cases} i^2, & i=0,1,2 \\ (k+i-4)(v-k+i-5), & i=3,4 \end{cases}$$

and

$$\beta_i = \begin{cases} (k-i)(v-k-i), & i=0,1 \\ 2(v-k-2), & i=2 \\ (5-i)(4-i), & i=3,4 \end{cases}.$$

So the eigenvalues of the corresponding tridiagonal matrix B are $\tau_0 = (k-1)(v-k)$,

$$\tau_1 = \frac{2k + W - 5 + X}{2}$$
$$\tau_2 = \frac{2k + W - 5 - X}{2}$$
$$\tau_3 = \frac{2k - W - 5 + Y}{2}$$
$$\tau_4 = \frac{2k - W - 5 - Y}{2}$$

where X and Y, given by

$$X^2 = v^2 + 4v - 11 - 4k(v-k) + 2W$$

and

$$Y^2 = v^2 + 4v - 11 - 4k(v-k) - 2W$$

are required to be integers. This rules out many, but not all, feasible parameter sets for perfect 2-codes in $J(v,k)$.

The best results to date on this problem are those of Etzion [14], who found designs in the local structure and obtained strong number-theoretic restrictions on the existence of such codes. The case $e = 1$ seems simplest and this is worth revisiting.

8 Simple subsets

Let $(X, \mathcal{A})$ be a symmetric association scheme and $C \subseteq X$. The outer distribution matrix of C is the $|X| \times (d+1)$ matrix D with rows indexed by the vertices, columns indexed by the relations and (y,i) entry equal to the number of elements of C which are i-related to y. The rank of D is $s^* + 1$ where s^* is the dual degree of C. Clearly D has at least $s^* + 1$ distinct rows. If equality holds then, following [21], we say that C is a *simple subset*. In [12] and in [21], simple subsets play an important role in the study of quotient schemes. One may think of completely regular codes as the "P-polynomial" simple subsets. Examples are known of such P-polynomial simple subsets in non-P-polynomial association schemes. More specifically, as one would naturally expect, a non-P-polynomial association scheme may have a P-polynomial quotient. (For example, one may obtain cycles from dihedral groups.) There should be more interesting examples of this sort.

On the other hand, we know very little about "Q-polynomial" simple subsets. If C is such a subset and the eigenspaces are ordered so that $E_j\mathbf{x} \neq 0$ for $j = 0, \ldots, s^*$, then the span of the vectors $\{E_0\mathbf{x}, \ldots, E_{s^*}\mathbf{x}\}$ is closed under entrywise multiplication (equitable partition!). So there exist coefficients $q_{i,j}^k$ satisfying

$$(E_i\mathbf{x}) \circ (E_j\mathbf{x}) = \sum_{k=0}^{s^*} q_{i,j}^k E_k\mathbf{x}.$$

We know of completely regular codes for which these analogues of Krein parameters can be negative. So one asks for sufficient conditions to ensure $q_{i,j}^k \geq 0$ for all i, j and k. It is fairly clear what the Q-polynomial property should be (except that the reader may object to the possibility that the Q-polynomial ordering for C may be inconsistent with the Q-polynomial ordering for $(X, \mathcal{A})$). We have the analogue of a completely regular partition for general association schemes and these Q-polynomial simple codes might provide a way to obtain some Q-polynomial quotients. We do currently have quite a shortage of Q-polynomial schemes that are not P-polynomial.

It is sometimes possible to assign a meaningful partial order on the eigenspaces of an association scheme which is not Q-polynomial. For example, the eigenspaces of the symmetric group S_n are indexed by all partitions of n and these are naturally ordered by reverse dominance order. We can thus define the dual width of a set of permutations as the largest height of a partition appearing in its dual degree set. It seems fruitful to study the extremal codes and the simple subsets in this association scheme.

9 Summary

I have given the opinion that completely regular codes are unlikely to be of further use to coding theorists. Recent record-breaking codes have been found by augmenting known codes with nice structure by messy collections of additional codewords found using clique-finding software. Furthermore, many coding theorists have lost interest in what was once called the "Main Problem of Coding Theory": that of finding the largest binary code for a given length and minimum distance. Current interest is rather directed toward turbo codes, LDPC codes, and convolutional codes. Even without a proof of Neumaier's conjecture, the search for large completely regular codes with strong error-correcting capabilities seems less than promising.

On the other hand, completely regular codes seem to play a fundamental role in the study of distance-regular graphs and there are many structural questions begging to be answered. Codes of width $d - s^*$ appear in regular semilattices, completely regular partitions are intrinsic to the study of distance-regular quotient graphs, such codes may play a role in uniqueness proofs, and such codes sometimes give a useful spanning set for a direct sum of eigenspaces of a Q-polynomial graph.

Finally, let me observe that the bound $d(C) \leq 7$ conjectured by Neumaier for the Hamming graphs (with the exception of the binary repetition codes and the extended binary Golay code) may in fact hold for completely regular codes in arbitrary distance-regular graphs. Aside from codes contained in antipodal classes, I know of no completely regular code in any distance-regular graph with $d(C) > 8$.

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