The biweight enumerator and the subconstituent algebra of the n-cube

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Abstract. Let C be a binary code of length n. For a four-tuple $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of non-negative integers summing to n, let

$$\ell_\alpha = \left|\left\{(c, c') \in C \times C : \mathsf{wt}(c) = \alpha_2 + \alpha_3, \partial(c, c') = \alpha_1 + \alpha_3, \mathsf{wt}(c') = \alpha_1 + \alpha_2\right\}\right|.$$

We study the biweight enumerator

$$\mathcal{W}_C(y_0, y_1, y_2, y_3) = \sum_{\alpha=(\boldsymbol{\alpha_0},\boldsymbol{\alpha_1},\boldsymbol{\alpha_2},\boldsymbol{\alpha_3})} \ell_\alpha \; y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}.$$

For binary linear codes, MacWilliams identities for this enumerator were given by MacWilliams, et al. in 1972 [4]. We give a new proof of these identities using the Terwilliger algebra of the n-cube. Further, we consider some families of non-linear codes whose dual biweight enumerators necssarily have non-negative coefficients. For unrestricted codes, a characterization of the positive semidefinite cone of the Terwilliger algebra is given which leads to new inequalities for the coefficients of the biweight enumerator of an unrestricted code. Each extremal ray of the positive semidefinite cone corresponds to a projection onto some irreducible module of the S_n action on the free real vector space over the binary n-tuples. It remains to find computationally useful formulae for these inequalities.

1 Introduction

Let n be a positive integer and let $X = \{0, 1\}^n$. The *Hamming weight* of $u \in X$, denoted as $\mathsf{wt}(u)$, is the number of non-zero coordinates in the tuple u. For $u, v \in X$, the *Hamming distance*, $\partial(u, v)$, is defined as the number of coordinates h $(1 \le h \le n)$ with $u_h \ne v_h$. By a *(binary) code* C of length n, we simply mean any non-trivial subset of X. In this paper, we will always assume that C contains the zero tuple $\mathbf{0} = 000 \cdots 0$.

Any three (not necessarily distinct) tuples v_1, v_2, v_3 from X give rise to a triple of non-negative integers (i, j, k) via the equations

$$\partial(v_2, v_3) = i, \qquad \partial(v_1, v_3) = j, \qquad \partial(v_1, v_2) = k.$$

Not every triple (i,j,k) of non-negative integers occurs in this way; the triangle inequality must hold, none of i,j,k may exceed n, their sum must not exceed $2n$, and $i+j+k$ must be even. For n fixed, there is a simple bijection between the set of all such triples (i,j,k) and the set of all ordered quadruples $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of non-negative integers summing to n: it is given by the system

$$\begin{aligned} \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 &= n, \\ \alpha_2 + \alpha_3 &= i, \\ \alpha_1 + \qquad \alpha_3 &= j, \\ \alpha_1 + \alpha_2 \qquad &= k. \end{aligned}$$

(In fact, for any three vertices v_1, v_2, v_3 in the n-cube whose pairwise distances are as above, there exists a unique vertex $u \in X$ with $\partial(u, v_i) = \alpha_i$ for $i = 1, 2, 3$.) Henceforth, we will use $\vartheta(i,j,k) = \alpha$ to indicate that the triple (i,j,k) is related to the quadruple $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ via these equations. By a slight abuse of terminology, we will refer to α as a *composition of n (into four parts)*.

Let C be a binary code of length n. For each composition $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ of n into four parts, let

$$\ell_\alpha = |\{(c, c') \in C \times C : \mathsf{wt}(c) = i,\ \partial(c, c') = j,\ \mathsf{wt}(c') = k\}|$$

where $(i,j,k) = \vartheta^{-1}(\alpha)$. Consider the enumerator

$$\mathcal{W}(y_0, y_1, y_2, y_3) = \sum_\alpha \ell_\alpha y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3}.$$

Below, we will establish a connection between this enumerator for a linear code and the enumerator for its dual. Moreover, we will examine a linear programming approach based on this enumerator. First, we will need to compute a change-of-basis matrix for an important matrix algebra related to our problem.

2 Two bases for the subconstituent algebra $\mathbb{T}$

We consider the binary Hamming graph Q_n with vertex set X and adjacency matrices $A_0, \ldots, A_n$. The primitive idempotents for the Bose-Mesner algebra $\mathbb{A}$ will be denoted $E_0, \ldots, E_n$. We have

$$E_j = \frac{1}{2^n} \sum_{i=0}^{n} Q_{i,j} A_i$$

where

$$Q_{i,j} = [z^j](1+z)^{n-i}(1-z)^i. \tag{1}$$

Here, as below, we use this notation to describe the coefficient of z^j in the expansion of the (finite) power series $(1+z)^{n-i}(1-z)^i$.

In order to construct the dual Bose-Mesner algebra $\mathbb{A}^*$, we first select a base point: here we choose the tuple $\mathbf{0}$. We define A_i^* to be the diagonal matrix with

(x,x)-entry equal to $|X|(E_i)_{x,\mathbf{0}}$ and we define E_j^* to be the diagonal matrix with (x,x)-entry equal to $(A_j)_{x,\mathbf{0}}$. These give us two bases for $\mathbb{A}^*$. The subconstituent algebra (or, Terwilliger algebra) of the n-cube is then the algebra generated by $\mathbb{A}$ and $\mathbb{A}^*$. We denote this non-commutative algebra by $\mathbb{T}$. Terwilliger algebras can be defined for any association scheme and these algebras have been proposed in the analysis of Q-polynomial distance regular graphs and more broadly. But the Terwilliger algebras of the n-cubes are particularly well-behaved and are the focus of several articles.

For a composition α of n, we define matrices

$$L_\alpha = E_i^* A_j E_k^*$$

and

$$M_\alpha = E_i A_j^* E_k$$

where $\vartheta(i,j,k)=\alpha$. It is easy to see that $L_\alpha \neq 0$ if and only if the intersection number $p_{i,j}^k > 0$ and it is well-known $M_\alpha \neq 0$ if and only if the Krein parameter $q_{i,j}^k$ is non-zero. Our goal in this section is to describe the change-of-basis matrix from the basis

$$\mathcal{A} = \left\{ L_\alpha = E_i^* A_j E_k^* : p_{i,j}^k > 0, \vartheta(i,j,k) = \alpha \right\}$$

to the basis

$$\mathcal{B} = \left\{ M_\beta = E_r A_s^* E_t : q_{r,s}^t > 0, \vartheta(r,s,t) = \beta \right\}.$$

We freely use basic facts about this algebra (see [6–8, 3] for a full treatment).

It is now easy to compute the dimension of $\mathbb{T}$. From the previous section, we know that the triples (i,j,k) for which $p_{i,j}^k > 0$ are in one-to-one correspondence with compositions α of n into four parts. There are $\binom{n+3}{3}$ such compositions. So the dimension of $\mathbb{T}$ is $\binom{n+3}{3}$.

2.1 Change-of-basis coefficients

From above, we know that there are unique rational numbers t_α^β which satisfy the following system of equations

$$E_r A_s^* E_t = \frac{1}{2^n} \sum_\alpha t_\alpha^\beta E_i^* A_j E_k^*$$

where $\vartheta(i,j,k) = \alpha$ and $\vartheta(r,s,t) = \beta$. We wish to learn more about these coefficients t_α^β.

Note that the dimension of the Terwilliger algebra $\mathbb{T}$ is also the dimension of the vector space $\mathsf{Hom}_n(y_0,y_1,y_2,y_3)$ of homogeneous polynomials of degree n in four variables. In fact, we have a natural isomorphism of vector spaces $\varphi : \mathbb{T} \to \mathsf{Hom}_n(y_0,y_1,y_2,y_3)$ given by

$$\varphi : E_i^* A_j E_k^* \mapsto y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3} \qquad (\vartheta(i,j,k)=\alpha).$$

This vector space isomorphism can be exploited to solve our problem in an elegant way. For the n-cube, it so happens that the Terwilliger algebra $\mathbb{T}_n$ is coherent. That is, we have a coherent configuration $(X, \mathcal{R})$ with relations corresponding to the zero-one basis

$$\left\{L_\alpha = E_i^* A_j E_k^* : \; p_{i,j}^k > 0\right\}. \tag{2}$$

These matrices can be extracted from a generating function as follows. Let Z_0, Z_1, Z_2, Z_3 be four commuting indeterminates and consider the $2^n \times 2^n$ matrix

$$\begin{aligned} \Phi_n &= \begin{pmatrix} Z_0 & Z_1 \\ Z_3 & Z_2 \end{pmatrix}^{\otimes n} && (3) \\ &= \left(Z_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + Z_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + Z_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + Z_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)^{\otimes n}. && (4) \end{aligned}$$

If $\vartheta(i, j, k) = \alpha$ (with n pre-specified), we have

$$E_i^* A_j E_k^* = [Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}] \; \Phi_n. \tag{5}$$

In particular, the standard basis is our basis of Schur idempotents for $\mathbb{T}_1$.

Our second distinguished basis for $\mathbb{T}_1$ is

$$\{E_0 A_0^* E_0, E_0 A_1^* E_1, E_1 A_0^* E_1, E_1 A_1^* E_0\}, \tag{6}$$

explicitly

$$\left\{ \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right\}. \tag{7}$$

The change-of-basis matrix for $\mathbb{T}_1$ from the standard basis (2) to the basis (7) is given by

$$H = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}. \tag{8}$$

This allows us to recover the coefficients t_α^β. If we define

$$\Psi_n = \left(Z_0(E_0 A_0^* E_0) + Z_1(E_0 A_1^* E_1) + Z_2(E_1 A_0^* E_1) + Z_3(E_1 A_1^* E_0)\right)^{\otimes n}, \tag{9}$$

then

$$M_\beta = E_r A_s^* E_t = [Z_0^{\beta_0} Z_1^{\beta_1} Z_2^{\beta_2} Z_3^{\beta_3}] \; \Psi_n \tag{10}$$

where $\vartheta(r, s, t) = \beta$. Using (8), we may write

$$\begin{aligned} \Psi_n = \frac{1}{2^n} \big(& (Z_0 + Z_1 + Z_2 + Z_3)(E_0^* A_0 E_0^*) + \\ & (Z_0 - Z_1 - Z_2 + Z_3)(E_1^* A_1 E_0^*) + \\ & (Z_0 - Z_1 + Z_2 - Z_3)(E_1^* A_0 E_1^*) + \\ & (Z_0 + Z_1 - Z_2 - Z_3)(E_0^* A_1 E_1^*) \big)^{\otimes n}. \end{aligned} \tag{11}$$

Thus, using (5) and (10), we have the following theorem:

Theorem 1. *The connection coefficients t_α^β satisfying*

$$M_\beta = \frac{1}{2^n} \sum_\alpha t_\alpha^\beta L_\alpha$$

are given by

$$t_\alpha^\beta = [Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}] \left(Z_0+Z_1+Z_2+Z_3\right)^{\beta_0} \left(Z_0-Z_1-Z_2+Z_3\right)^{\beta_1} \cdot \\ \left(Z_0-Z_1+Z_2-Z_3\right)^{\beta_2} \left(Z_0+Z_1-Z_2-Z_3\right)^{\beta_3}. \tag{12}$$

Proof. We use the duality of the algebra to fill in the gaps above. Let $S = 2^{-n/2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes n}$. Then S diagonalizes the Bose-Mesner algebra. So

$$S^T A_i S = A_i^*, \qquad S^T E_j S = E_j^*.$$

Since S is symmetric, we also have

$$S^T A_i^* S = A_i, \qquad S^T E_j^* S = E_j$$

so that $S^T L_\alpha S = M_\alpha$ and vice versa. Since this holds for $n = 1$, we find $S^T \Phi_n S = \Psi_n$ and $S^T \Psi_n S = \Phi_n$ for all n. Now the calculation follows. □

2.2 Symmetry

It is well-known that the association scheme of the n-cube Q_n is self-dual. (The explicit duality operator was presented in the previous proof.) For any indices i, j and k, we have

$$p_{i,j}^k = q_{i,j}^k$$

wherever these quantities are defined. In addition, we have the usual symmetry observed in association schemes,

$$n_k p_{i,j}^k = n_j p_{i,k}^j$$

where $n_k = \binom{n}{k}$ is the k^{th} valency of Q_n. In fact, if $\vartheta(i,j,k) = \alpha$, then we have

$$n_k p_{i,j}^k = \frac{n!}{\alpha_0!\alpha_1!\alpha_2!\alpha_3!}.$$

Given Theorem 1, it is now clear that we have

$$\frac{t_\alpha^\beta}{\alpha_0!\alpha_1!\alpha_2!\alpha_3!} = \frac{t_\beta^\alpha}{\beta_0!\beta_1!\beta_2!\beta_3!}.$$

3 Linear Programming

Let us briefly review Delsarte's linear programming bound for codes. We seek an upper bound on the size of a code $C \subseteq X$ which has minimum distance at least d. We find that $|C|$ is bounded above by

$$\begin{aligned} &\textsf{maximize} \sum_{i=0}^{n} a_i \\ &\textsf{subject to} \\ &\qquad \sum_{i=0}^{n} Q_{i,j} a_i \geq 0 \qquad (0 \leq j \leq n) \\ &\qquad a_i = 0 \qquad (1 \leq i < d) \\ &\qquad a_0 = 1 \\ &\qquad a_i \geq 0 \qquad (d \leq i \leq n). \end{aligned}$$

This is derived as follows. Given a non-empty code C with minimum distance at least d and characteristic vector χ, define

$$a_i = \frac{1}{|C|} \chi^T A_i \chi, \qquad (0 \leq i \leq n).$$

Then, clearly, $a_0 = 1$, each $a_i \geq 0$ and $\sum_i a_i = |C|$. Now define

$$b_j = \frac{|X|}{|C|} \chi^T E_j \chi, \qquad (0 \leq j \leq n).$$

Since each E_j is symmetric and positive semi-definite, we know that each $b_j \geq 0$. Moreover, since

$$E_j = \frac{1}{|X|} \sum_{i=0}^{n} Q_{i,j} A_i,$$

we have $b_j = \sum_i Q_{i,j} a_i$.

We now extend this approach to the Terwilliger algebra of the n-cube. Throughout, we assume that our code contains the zero word.

For each triple (i, j, k) such that $p^j_{i,k} > 0$, let $\alpha = \vartheta(i, j, k)$ and define

$$\ell_\alpha = \chi^T E^*_i A_j E^*_k \chi.$$

Then

$$\ell_\alpha = |\{(y, z) \in C \times C : \mathsf{wt}(y) = i,\ \mathsf{wt}(z) = k,\ \partial(y, z) = j\}|\,.$$

Thus ℓ_α is a non-negative integer. As well, the sum of all such ℓ_α is equal to $|C|^2$. These quantities seem deserving of further study. Similarly, for each triple (r, s, t) such that $q^s_{r,t} > 0$, we set $\beta = \vartheta(r, s, t)$ and introduce

$$m_\beta = |X| \cdot \chi^T E_r A^*_s E_t \chi.$$

Since

$$M_\beta = \frac{1}{2^n}\sum_\alpha t_\alpha^\beta L_\alpha,$$

we have

$$m_\beta = \sum_\alpha t_\alpha^\beta \ell_\alpha.$$

It remains to present an efficient strategy for bounding (or interpreting) the parameters m_β of a code. We will first deal with the case of linear codes.

3.1 MacWilliams identities

Assume C is a binary linear code. Let $C_j^\perp$ denote the set of dual codewords of C having weight j. A matrix diagonalizing the Bose-Mesner algebra is $2^{-n/2}S$ where

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes n}$$

has (x,y)-entry $(-1)^{x\cdot y}$ (here $\cdot$ denotes the binary dot product). Let S_j denote the submatrix of S obtained by restricting to columns indexed by tuples of weight j. Then

$$E_j = \frac{1}{2^n}S_jS_j^T$$

for $0 \le j \le d$. For $x \in X$, we use s_x to denote the column of S indexed by x.

We use the following standard fact:

$$\sum_{x\in C}(-1)^{y\cdot x} = \begin{cases} |C|, & \text{if } y \in C^\perp; \\ 0, & \text{otherwise.} \end{cases}$$

So

$$u_j(C) := E_j\chi = \frac{1}{2^n}S_jS_j^T\chi = \frac{|C|}{2^n}\sum_{y\in C_j^\perp} s_y,$$

and if C is an $[n,k,d]$-code, this gives

$$u_j(C) := 2^{k-n}\sum_{y\in C_j^\perp} s_y.$$

Now if $\beta = \vartheta(i,j,k)$,

$$m_\beta = 2^n u_i(C)^T A_j^* u_k(C) = 4^n\langle u_i(C), u_j(0)\circ u_k(C)\rangle = 4^n\langle u_j(0), u_i(C)\circ u_k(C)\rangle$$

where $u_j(0)$ denotes the column of E_j indexed by 0. Clearly

$$u_j(0) = \frac{1}{2^n}\sum_{\mathsf{wt}(w)=j} s_w.$$

We need to examine the Schur product of $u_i(C)$ with $u_k(C)$. We have

$$u_i(C) \circ u_k(C) = 4^{k-n} \sum_{y \in C_i^\perp} \sum_{z \in C_k^\perp} s_y \circ s_z = 4^{k-n} \sum_{y \in C_i^\perp} \sum_{z \in C_k^\perp} s_{y \oplus z}$$

where $\oplus$ denotes vector addition over $GF(2)$.

Now we have

$$\begin{aligned} m_\beta &= 2^n \chi^T E_i A_j^* E_k \chi \\ &= 4^n \langle u_j(0), u_i(C) \circ u_k(C) \rangle \\ &= 4^k \sum_{y \in C_i^\perp} \sum_{z \in C_k^\perp} \langle u_j(0), s_{y \oplus z} \rangle \\ &= 2^{2k-n} \sum_{y \in C_i^\perp} \sum_{z \in C_k^\perp} \sum_{\mathsf{wt}(w)=j} \langle s_w, s_{y \oplus z} \rangle. \end{aligned}$$

Now the columns of S are pairwise orthogonal; in fact, $S^T S = 2^n I$. So

$$m_\beta = 2^n \chi^T E_i A_j^* E_k \chi = 4^k \left| \left\{ (y,z) \in C_i^\perp \times C_k^\perp : \mathsf{wt}(y \oplus z) = j \right\} \right|.$$

Written a bit differently, this is

$$m_\beta = 2^{2k} \left| \left\{ (y,z) \in C^\perp \times C^\perp : \mathsf{wt}(y) = i,\ \partial(y,z) = j,\ \mathsf{wt}(z) = k \right\} \right|.$$

Now, using the results of Section 2.1, m_β is the coefficient of $y_0^{\beta_0} y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3}$ in the expansion of $\mathcal{W}_C(H\mathbf{y})$ where H is as above and $\mathbf{y} = [y_0, y_1, y_2, y_3]^T$. Thus we have a new proof of

Theorem 2 (MacWilliams' identities for the biweight enumerator [4]).

$$\mathcal{W}_{C^\perp}(y_0, y_1, y_2, y_3) = \frac{1}{|C|^2} \cdot \tag{13}$$
$$\mathcal{W}_C(y_0 + y_1 + y_2 + y_3, y_0 - y_1 - y_2 + y_3, y_0 - y_1 + y_2 - y_3, y_0 + y_1 - y_2 - y_3)$$

Of course, such a theorem always gives us a linear programming bound as a by-product.

Corollary 1. *If C is a linear code, then, for each triple (i,j,k) with $q_{i,j}^k > 0$,*

$$m_\beta \geq 0.$$

3.2 Krein conditions

We note that the inequalities $m_\beta \geq 0$ do not hold for general codes. For example, for the code $C = \{000, 001, 011\}$ in Q_3, we have $m_\beta = -\frac{2}{3}$ for $\beta = \vartheta(0,1,3)$ and for $\beta = \vartheta(1,2,3)$.

Let us assume that code C is distance-invariant: there exist integers $a_0, a_1, \ldots, a_n$ such that, for any $c \in C$,

$$|\{c' \in C : \partial(c, c') = i\}| = a_i.$$

Suppose $\beta = \vartheta(i, j, k)$. We know that

$$m_\beta = 2^n u_i(C)^T A_j^* u_k(C) = 4^n \langle u_j(0), u_i(C) \circ u_k(C) \rangle.$$

Since C is distance-invariant and $u_j(C) = \sum_{c \in C} u_j(c)$, we have

$$m_\beta = \frac{2^n}{|C|} \langle u_j(C), u_i(C) \circ u_k(C) \rangle.$$

Definition: Let C be a q-ary code of length n with characteristic vector χ and let E_j denote the j^{th} primitive idempotent of the Hamming scheme $H(n, q)$. The *Krein parameters* $\bar{q}_{i,j}^k(C)$ of code C are given by

$$\bar{q}_{i,j}^k(C) = \|E_k \chi\| \cdot \langle E_k \chi, (E_i \chi) \circ (E_j \chi) \rangle.$$

We say C satisfies the *Krein conditions* provided all of its Krein parameters are non-negative.

For instance, from above, we see that every binary linear code satisfies the Krein conditions. Some completely regular codes fail. An easy example is any completely regular code of covering radius one with $|C| > \frac{1}{2} q^n$. However, if there is a completely regular partition [1, p351] of $H(n, q)$ with all cells having the same parameters of C, then we have a generalized coset graph and the Krein conditions for C follow from those of this graph, which is guaranteed to be distance-regular.

Theorem 3. *If C is a binary code of length n which is distance-invariant and satisfies the Krein conditions, then $m_\beta \geq 0$ for all compositions β of n.*

3.3 Nonlinear codes

Can we say anything more about the "dual" biweight enumerator of a non-linear binary code C? Assuming C is distance-invariant, we still have some control on the values m_β as the following proposition shows.

Proposition 1. *For any non-trivial code C with biweight enumerator*

$$\mathcal{W}(y_0, y_1, y_2, y_3) = \sum_\alpha \ell_\alpha \; y_0^{\alpha_0} y_1^{\alpha_1} y_2^{\alpha_2} y_3^{\alpha_3},$$

the transform $\mathcal{W}(H\mathbf{y}) = \sum_\beta m_\beta y_0^{\beta_0} y_1^{\beta_1} y_2^{\beta_2} y_3^{\beta_3}$ satisfies

$$\sum_{i=0}^{n} m_{\vartheta(i,j,k)} \geq 0$$

and

$$\sum_{j=0}^{n} m_{\vartheta(i,j,k)} \geq 0.$$

Note: Since each composition β corresponds to a unique triple (i,j,k), we are summing over all triples (i,j,k) with fixed values of j and k in the first instance and with fixed values of i and k in the second. By symmetry, there is also such an inequality for the sum of m_β where i and j are fixed and k varies.

Proof. Suppose $\beta = \vartheta(i,j,k)$. Then, we have

$$m_\beta = 2^n \chi^T E_i A_j^* E_k \chi. \tag{14}$$

Thus we find

$$\sum_{\substack{\beta_2+\beta_3=i \\ \beta_1+\beta_2=k}} m_\beta = \sum_{j=0}^{n} 2^n \chi^T E_i A_j^* E_k \chi = 4^n \chi^T E_i E_0^* E_k \chi$$

which can be written

$$4^n \langle u_i(C), \mathbf{e}_0 \circ u_k(C) \rangle = 4^n \langle \mathbf{e}_0, u_i(C) \circ u_k(C) \rangle.$$

Now if we assume C is distance-invariant, then for $v \in C$ the v-entry of $u_h(C)$ is independent of v. Moreover, since

$$\langle \chi, u_h(C) \rangle = \sum_{i=0}^{n} \langle u_i(C), u_h(C) \rangle = \langle u_h(C), u_h(C) \rangle,$$

this value $(u_h(C))_v$ is never negative. We are assuming $0 \in C$, so

$$\sum_{\substack{\beta_2+\beta_3=i \\ \beta_1+\beta_2=k}} m_\beta = (u_i(C))_0 \cdot (u_k(C))_0 \geq 0$$

with equality if and only if either $u_i(C) = 0$ or $u_k(C) = 0$.

Similarly,

$$\sum_{\substack{\beta_1+\beta_3=j \\ \beta_1+\beta_2=k}} m_\beta \geq 0$$

since

$$\sum_{\beta} m_\beta = \sum_{i=0}^{n} 2^n \chi^T E_i A_j^* E_k \chi = 2^n \chi^T A_j^* E_k \chi.$$

But

$$\chi^T A_j^* E_k \chi = \langle \chi, u_j(0) \circ u_k(C) \rangle = \langle u_j(0), \chi \circ u_k(C) \rangle = \eta \langle u_j(0), \chi \rangle$$

for some $\eta \geq 0$ since C is assumed to be distance-invariant. Finally, we observe that this last expression simplifies to

$$\eta \langle u_j(0), u_j(C) \rangle$$

which is non-negative since $0 \in C$ by hypothesis. $\square$

4 Positive semidefinite matrices in the Terwilliger algbra

The Terwilliger algebra $\mathbb{T}_n$ can be described as follows. Of course

$$\mathsf{Mat}_2(\mathbb{R})^{\otimes n} \cong \mathsf{Mat}_{2^n}(\mathbb{R})$$

and we treat these as the same algebra. The symmetric group S_n acts on tensor products via

$$(A_1 \otimes \cdots \otimes A_n)^\sigma = A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)}.$$

This extends linearly to $\mathsf{Mat}_2(\mathbb{R})^{\otimes n}$. We may take $\mathbb{T}_n$ to be the subalgebra consisting of those matrices fixed by each σ in S_n.

It is easy to see that a basis matrix L_α with $\alpha = \vartheta(i,j,k)$ is symmetric if and only if $i = k$, i.e., if $\alpha_1 = \alpha_3$. The number of such α is clearly equal to the number of triples $(\gamma_0, \gamma_1, \gamma_2)$ of non-negative integers summing to n in which γ_1 is even. This, in turn, is the same as the number of pairs from an $(n+2)$-element set whose smaller member is odd. This is given by

$$r_n := (n+1) + (n-1) + (n-3) + \cdots = (\lfloor \frac{n}{2} \rfloor + 1)(\lfloor \frac{n+1}{2} \rfloor + 1),$$

or

$$r_n = \begin{cases} \frac{(n+2)^2}{4}, & \text{if } n \text{ even;} \\ \frac{(n+1)(n+3)}{4}, & \text{if } n \text{ odd.} \end{cases}$$

So the subspace of symmetric matrices in $\mathbb{T}_n$ has dimension

$$\frac{1}{2} \cdot \left[\binom{n+3}{3} + r_n \right] = \begin{cases} \frac{(n+2)(n+4)(2n+3)}{24}, & \text{if } n \text{ even;} \\ \frac{(n+1)(n+3)(2n+7)}{24}, & \text{if } n \text{ odd.} \end{cases}$$

It is clear that the positive semidefinite matrices in $\mathbb{T}_n$ form a cone within this subspace of symmetric matrices. For if E and F are positive semidefinite and $c, d \geq 0$, then for any vector x we have $x^T E x \geq 0$ and $x^T F x \geq 0$ giving

$$x^T(cE + dF)x = cx^T E x + dx^T F x \geq 0.$$

Henceforth denote this cone by $\mathcal{C}_T$.

Our next goal is to describe the positive semidefinite cone $\mathcal{C}_T$ of the algebra $\mathbb{T}_n$.

5 A symmetrized torus

Before looking at the extreme rays of this cone, we explore an interesting subcone with elementary structure.

We can find samples of p.s.d. matrices as follows. Let $u_1, \ldots, u_n$ be unit vectors in $\mathbb{R}^2$. Let $G_i = u_i u_i^T$. Then each G_i is a rank one projection onto the span of u_i and

$$G = G_1 \otimes \cdots \otimes G_n$$

is a rank one projection onto the span of the vector

$$u_1 \otimes \cdots \otimes u_n.$$

Now sum all the matrices in the S_n orbit of G to obtain a positive semidefinite matrix in $\mathbb{T}_n$. Denote the cone generated by these matrices by $\mathcal{C}_0$.

The rank one projections in $\mathsf{Mat}_2(\mathbb{R})$ are naturally topologized as projective one-space, this being homeomorphic to the unit circle. The foregoing discussion leads us to seek out the topological structure of the space

$$(S^1 \times \cdots \times S^1)/S_n.$$

This is described as follows. We consider the torus $\times_1^n S^1$ with its product topology (or, equivalently, the topology induced from the usual topology on $\mathbb{R}^{2^n}$). We have a natural equivalence relation under the coordinatewise action of the symmetric group S_n. We would like to describe the quotient space.

Problem: Determine the homotopy type (or, at least, the homology) of the symmetrized cartesian product $(\times_1^n S^1)/S_n$.

One can easily answer this question in the case $n = 2$. We wish to take a quotient of the ordinary torus $S^1 \times S^1$ over the two element group acting on the coordinates. With a few cut-and-paste diagrams, we convince ourselves that this space is homeomorphic to a Möbius strip. So this subset of the positive semidefinite cone of $\mathbb{T}_2$ is somehow a pointed cone over a Möbius strip embedded in $\mathbb{R}^8$. But this is only $\mathcal{C}_0$; is this the structure of $\mathcal{C}_T$?

It seems possible to describe the topology of this set of matrices $\times_1^n S^1/S_n$ as a CW-complex. There is a single n-cell which can be viewed as the set

$$C_n = \{(\theta_1, \ldots, \theta_n) : 0 < \theta_1 < \cdots < \theta_n < \pi\}.$$

The various faces are obtained by replacing any subset of the strict inequalities "$<$" with weak inequalities "$\leq$". The gluing together of these cells seems complicated.

By way of comparison, the positive semidefinite cone of $\mathsf{Mat}_k(\mathbb{R})$ has each extreme ray generated by a rank one projection operator. These are in one-to-one correspondence with lines through the origin in $\mathbb{R}^k$. So the boundary of the positive semidefinite cone is a pointed cone over projective $(k-1)$-space $\mathbb{P}^{k-1}$.

6 Ranks

If $G_1 = u_1 u_1^T$ and $G_2 = u_2 u_2^T$, then

$$G = (G_1 \otimes G_2) + (G_2 \otimes G_1)$$

has eigenvalues

$$0,\ 0,\ 1 + \langle u_1, u_2 \rangle^2,\ \langle u_1, v \rangle^2$$

where v is a unit vector orthogonal to u_2. So the matrix G has rank two except when $u_1 = \pm u_2$, in which case $\mathsf{rank}(G) = 1$. In terms of our topological description, the matrices in the interior of the Möbius strip have rank two and those on the boundary have rank one.

If $G_1 = G_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $G_3 = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}$ with $x^2 + y^2 = 1$, then G has eigenvalues

$$0,\ 0,\ 0,\ 0,\ 0,\ 4x^2 + 2,\ 2y^2,\ 2y^2.$$

If $G_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $G_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $G_3 = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}$ with $x^2 + y^2 = 1$, then G has eigenvalues

$$0,\ 0,\ 0,\ 2,\ 1 + \gamma,\ 1 - \gamma$$

(with each of the latter two eigenvalues appearing with multiplicity two) where $\gamma = |x^2 - y^2|$.

If, in the above special case, we replace G_2 by $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then we obtain an 8×8 matrix G with eigenvalues

$$0,\ 0,\ 0,\ 2x^2 + (x + y)^2 + 1,\ \frac{1}{2}(x^2 - xy + 2y^2 + \sqrt{\gamma}), \frac{1}{2}(x^2 - xy + 2y^2 - \sqrt{\gamma})$$

where

$$\gamma = 1 - 2xy|x^2 - y^2|.$$

If $G_1 = uu^T$, $G_2 = vv^T$ and $G_3 = ww^T$ where

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad v = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \qquad w = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

with $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1$, then G has eigenvalues

$$0,\ 0,\ 0,\ 2 + 4\langle u, v\rangle\langle u, w\rangle\langle v, w\rangle,$$

$$2 - \langle u, v\rangle\langle u, w\rangle\langle v, w\rangle \pm \Gamma$$

(with each of the latter two eigenvalues appearing with multiplicity two) where $\Gamma^2 = x_1^4 y_2^4 + y_1^4 x_2^4 + y_1^4 y_2^4 + 4x_1 y_1^3 x_2 y_2^3 - 2x_1^3 y_1 x_2 y_2^3 - 2x_1 y_1^3 x_2^3 y_2 + 3x_1^2 y_1^2 x_2^2 y_2^2 - x_1^2 y_1^2 y_2^4 - y_1^4 x_2^2 y_2^2$.

Of course, when $n = 3$, G has trace equal to six.

In general, G has eigenvalue zero with multiplicity at least three, independent of the three vectors $u_1 = [x_1, y_1]$, $u_2 = [x_2, y_2]$ and $u_3 = [x_3, y_3]$. One of the three remaining eigenvalues is

$$\begin{aligned}\theta = {} & 2x_1^2 y_2^2 x_3^2 + 6y_1^2 y_2^2 y_3^2 + 2y_1^2 y_2^2 x_3^2 + 4y_1^2 x_2 y_2 x_3 y_3 \\ & + 2y_1^2 x_2^2 y_3^2 + 6x_1^2 x_2^2 x_3^2 + 2x_1^2 x_2^2 y_3^2 + 2x_1^2 y_2^2 y_3^2 \\ & + 4x_1^2 x_2 y_2 x_3 y_3 + 4x_1 y_1 x_2^2 x_3 y_3 + 4x_1 y_1 x_2 y_2 x_3^2 \\ & + 4x_1 y_1 x_2 y_2 y_3^2 + 4x_1 y_1 y_2^2 x_3 y_3 + 2y_1^2 x_2^2 x_3^2.\end{aligned}$$

The other two, each having multiplicity two, have less pleasant expressions. The rank is always at most five since we are summing the projections onto six one-dimensional spaces spanned by

$$u_1 \otimes u_2 \otimes u_3,\ u_1 \otimes u_3 \otimes u_2,\ u_2 \otimes u_1 \otimes u_3,\ u_2 \otimes u_3 \otimes u_1,\ u_3 \otimes u_1 \otimes u_2,\ u_3 \otimes u_2 \otimes u_1$$

which always satisfy the relation

$$u_1 \otimes u_2 \otimes u_3 - u_1 \otimes u_3 \otimes u_2 - u_2 \otimes u_1 \otimes u_3 + u_2 \otimes u_3 \otimes u_1 + u_3 \otimes u_1 \otimes u_2 - u_3 \otimes u_2 \otimes u_1 = 0.$$

Let $x_1, \ldots, x_n$ be chosen from the interval $[-1, 1]$ and let $y_i = \pm\sqrt{1 - x_i^2}$. Let $G_i = \begin{pmatrix} x_i^2 & x_i y_i \\ x_i y_i & y_i^2 \end{pmatrix}$ and let

$$G = \sum_{\sigma \in S_n} G_{\sigma(1)} \otimes \cdots \otimes G_{\sigma(n)}.$$

It should be possible to determine the entries of G and the eigenvalues of G as symmetric functions in the x_i. It would also be interesting to determine the rank of G in terms of the relative position of the vectors $\left\{ \begin{pmatrix} x_i \\ y_i \end{pmatrix} : 1 \leq i \leq n \right\}$. In general, the symmetrized matrix $G \in \mathbb{T}_n$ has trace $n!$ since it is the sum of $n!$ matrices each having trace one.

We now give a partial result regarding the rank of G.

Lemma 1. *For integers k and n, consider the collection of n-fold tensor products*

$$\{v_i := u_i \otimes \cdots \otimes u_i : 1 \leq i \leq k\}$$

where $u_1, \ldots, u_k$ are projectively distinct unit vectors in $\mathbb{R}^2$. The vectors v_i are linearly independent in $\mathbb{R}^{2^n}$ if and only if $k \leq n + 1$.

Proof. A vector of the form $u \otimes \cdots \otimes u$ has at most $n + 1$ distinct entries, the duplications being independent of the entries of u. So the vectors $\{v_i : i\}$ all lie in a space of dimension $n + 1$. Now assume $k \leq n + 1$ and suppose

$$c_1 v_1 + \cdots + c_k v_k = 0$$

in $\mathbb{R}^{2^n}$. If $u_i = [x_i, y_i]$, then we have $n + 1$ equations of the form

$$c_1 x_1^j y_1^{n-j} + \cdots + c_k x_k^j y_k^{n-j} = 0.$$

Applying an orthogonal transformation if necessary, we may assume that each $y_i \neq 0$. Then the above equations can be written

$$(c_1 y_1^n) \left(\frac{x_1}{y_1} \right)^j + \cdots + (c_k y_k^n) \left(\frac{x_k}{y_k} \right)^j = 0.$$

We recognize this as a Vandermonde system. Since the unit vectors u_i are projectively distinct, the ratios x_i / y_i are distinct real numbers so the only solution to this system is $c_i y_i^n = 0$ for all i. $\square$

This allows us to obtain an upper bound on the rank of G. Suppose for the moment that the u_i are distinct. Let w_i be a unit vector orthogonal to u_i. Then for any permutation $\sigma \in S_n$,

$$\langle w_i \otimes \cdots \otimes w_i, u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} \rangle = \langle w_i, u_{\sigma(1)} \rangle \cdots \langle w_i, u_{\sigma(n)} \rangle = 0$$

since $\langle w_i, u_i \rangle = 0$. Using the above lemma, this gives us n linearly independent vectors in the null space of G in the case where the u_i are projectively distinct. In the case where the u_i are not distinct, the rank of G will be smaller.

7 Inequalities for codes from cone $\mathcal{C}_0$

In order to obtain inequalities for unrestricted codes from the psd matrices studied in the previous section, we need only express each matrix G as a linear combination of the basis elements $L\alpha$.

Let's first look at $n = 3$. Let

$$u_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, u_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, u_3 = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix},$$

let

$$G_1 = u_1 u_1^T, G_2 = u_2 u_2^T, G_3 = u_3 u_3^T,$$

and let

$$G = G_1 \otimes G_2 \otimes G_3 + \cdots + G_3 \otimes G_2 \otimes G_1$$

(six terms). Then

$$G = \sum_\alpha s_\alpha L_\alpha$$

where s_α is a polynomial in $x_1, x_2, x_3, y_1, y_2, y_3$. It is easy to check that $\deg_{x_i} s_\alpha + \deg_{y_i} s_\alpha = 2$ for any i.

We now give the formula for s_α for arbitrary n. For a triple $\mu = (\mu_0, \mu_1, \mu_2)$ of nonnegative integers summing to n, let f_μ denote the monomial symmetric function of shape $1^{\mu_1} 2^{\mu_2}$ in variables $z_1, \ldots, z_n$; e.g., for $\mu = (1, 0, 2)$ and $n = 3$,

$$f_\mu = z_1^2 z_2^2 + z_1^2 z_3^2 + z_2^2 z_3^2.$$

Then, with $\mu_2 = \alpha_2$ and $\mu_1 = \alpha_1 + \alpha_3$,

$$s_\alpha = \mu_0! \mu_1! \mu_2! \cdot \prod_{i=1}^n x_i^2 f_\mu \left(\frac{y_1}{x_1}, \ldots, \frac{y_n}{x_n} \right).$$

The monomial symmetric function in variables $X_1, \ldots, X_n$ with shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ is the polynomial

$$m_\lambda(\underline{\mathrm{X}}) = \sum_{i_1, \ldots, i_k} X_{i_1}^{\lambda_1} \cdots X_{i_k}^{\lambda_k}$$

where the sum is over all injections $\{1, \ldots, k\} \to \{1, \ldots, n\}$ where j is mapped to i_j. By the notation $[2^k 1^\ell]$, we refer to the partition of $m = 2k + \ell$ having k parts equal to two and ℓ parts equal to one. Now define

$$m_\alpha(x_1, \ldots, x_n, y_1, \ldots, y_n) = y_1^2 \cdots y_n^2 \;\cdot\; m_{[2^{\alpha_0} 1^{\alpha_1 + \alpha_3}]}\left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right).$$

Then

$$s_\alpha = m_\alpha.$$

Theorem 4. *For any real numbers $\theta_1, \ldots, \theta_n$, we have*

$$\sum_\alpha s_\alpha \ell_\alpha \geq 0$$

where $x_i = \cos(\theta_i)$ and $y_i = \sin(\theta_i)$.

Proof. For these values of x_i and y_i, the matrix

$$M = \sum_\alpha s_\alpha L_\alpha$$

is a positive semidefinite matrix in $\mathbb{T}_n$. □

8 A family of commutative subalgebras

Observe that the (infinite) set of inequalities given by the theorem includes Delsarte's inequalities. For if we take $u_1 = \cdots = u_j = 2^{-1/2}[1, -1]^T$ and $u_{j+1} = \cdots = u_n = 2^{-1/2}[1, 1]^T$, then the symmetrized tensor product G is equal to the j^{th} primitive idempotent, E_j.

Likewise, if we choose $u_i = [0, 1]^T$ for j values of i and $u_i = [1, 0]^T$ for the remaining values, it is easy to see that $G = E_j^*$. So the primitive idempotents of $\mathbb{A}^*$ are also in the cone $\mathcal{C}_0$. In fact, there are an infinite number of commutative subalgebras of $\mathbb{T}_n$, each isomorphic to $\mathbb{A}$, all of whose primitive idempotents belong to $\mathcal{C}_0$.

We consider matrices G coming from the above construction from unit vectors $u_1, \ldots, u_n$ in $\mathbb{R}^2$ where the u_i take on at most two fixed distinct values. That is, for unit vectors $v, w \in \mathbb{R}^2$, consider the symmetrized tensor products

$$F_k = \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$$

where each $u_i = v$ for $n - k$ values of i and $u_i = w$ for k values of i. Let $\mathcal{U}_2(v, w)$ denote the subspace of $\mathbb{T}_n$ generated by these matrices.

Proposition 2. *Given the above definitions,*

– *if $v = \pm w$, then $\mathcal{U}_2(v, w)$ is a 1-dimensional subalgebra of $\mathbb{T}_n$;*

– *if* $v \perp w$*, then* $\mathcal{U}_2(v,w)$ *is an* $(n+1)$*-dimensional commutative subalgebra of* $\mathbb{T}_n$ *isomorphic to the Bose-Mesner algbra* $\mathbb{A}$*;*
– *If* w *is neither parallel nor orthogonal to* v*, then the matrices* F_k *do not commute and* $\mathcal{U}_2(v,w)$ *is not closed under multiplication.*

Proof. The first case is trivial. Now consider two symmetrized tensor products

$$F = \sum_\sigma u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$$

and

$$G = \sum_\pi v_{\pi(1)} \otimes \cdots \otimes v_{\pi(n)}.$$

We have

$$FG = \sum_\sigma \sum_\pi (u_{\sigma(1)} v_{\pi(1)}) \otimes \cdots \otimes (u_{\sigma(n)} v_{\pi(1)})$$

and

$$GF = \sum_\pi \sum_\sigma (v_{\pi(1)} u_{\sigma(1)}) \otimes \cdots \otimes (v_{\pi(1)} u_{\sigma(n)}).$$

Now if $u_i = w$ and $v_j = v$ then

$$(ww^T)(vv^T) = \langle w, v\rangle wv^T$$

while

$$(vv^T)(ww^T) = \langle w, v\rangle vw^T.$$

So F and G commute if and only if v is orthogonal to w or wv^T is a symmetric matrix. But this will happen if and only if $v = \pm w$. □

So, for each pair of orthogonal unit vectors in $\mathbb{R}^2$, we obtain a "copy" of the Bose-Mesner algebra with its primitive idempotents giving new inequalities for the biweight enumerator. This family of subalgebras interpolates between the Bose-Mesner algebra $\mathbb{A}$ and the dual Bose-Mesner algebra $\mathbb{A}^*$.

9 Irreducible S_n-modules

We know that every positive semidefinite matrix M inside $\mathbb{T}_n$ is diagonalizable. So we have

$$M = \sum_\theta \theta E_\theta$$

where the sum is over all (non-negative) eigenvalues θ of M. So the extreme rays of the positive semidefinite cone $\mathcal{C}_T$ are precisely those generated by these projection matrices E_θ.

Now since M lies in the commutant algebra of S_n, $E = E_\theta$ must be a projection onto an S_n-invariant subspace. Conversely, any such projection operator E lies in the cone $\mathcal{C}_T$. For if $\mathsf{colsp}E$ is S_n-invariant, then $\mathsf{nullsp}E$, its complement, is also S_n-invariant, So $E^\sigma = E$ for each $\sigma \in S_n$ and E belongs to $\mathbb{T}_n$. This proves

Lemma 2. *The extreme rays of the positive semidefinite cone are precisely the spans of all projection operators onto all irreducible S_n-submodules of $\mathbb{R}^{2^n}$.* □

Now we know the decomposition of $\mathbb{R}^{2^n}$ as an S_n module into irreducibles up to isomorphism. First, the subconstituents $\mathsf{im}E_k^*$ form an orthogonal decomposition. Then we observe that, for each k, we have $E_k^*\mathbb{T}E_k^*$ isomorphic to the Bose-Mesner algebra of the Johnson scheme $J(n,k)$. The action of S_n on this spaces decomposes into $k+1$ mutually non-isomorphic irreducible S_n modules, one for each partition $\lambda = (n-j, j)$ $(0 \le j \le k)$.

Thus, in the overall decomposition, the irreducible S_n module of isomorphism type indexed by the partition $\lambda = (n-j, j)$ appears with multiplicity $n+1-2j$ and has dimension $\binom{n}{j} - \binom{n}{j-1}$. Our interest, however, goes beyond this statistic. We seek expressions for the projections onto *each* of these irreducible S_n modules. For we know that each is expressible as a linear combination of the basis matrices L_α. To achieve this, we recall a bit of representation theory.

Schur's Lemma tells us that an S_n-homomorphism from any irreducible S_n module to any other irreducible is either an isomorphism or the zero map. Moreover, the only S_n-isomorphisms from an irreducible S_n module to itself are the (non-zero) multiples of the identity map. This essentially proves the following

Lemma 3. *Let the projection operator F be an extremal element of the positive semidefinite cone. Then F is the projection onto some irreducible S_n-submodule of V of isomorpism type $(n-j, j)$, say. This operator is uniquely determined by a set $\tau_j, \tau_{j+1}, \ldots, \tau_{n-j}$ of non-negative scalars satisfying $\sum_{h=j}^{n-j} \binom{n}{h} \tau_h = 1$. Specifically, the h^{th} diagonal block of the projection $E_h^* F E_h^*$ onto the h^{th} subconstituent is $\tau_h F_j^h$ where F_h^j is the j^{th} primitive idempotent in the standard Q-polynomial ordering for the Johnson scheme $J(n,h)$. The rank of F is $\binom{n}{j} - \binom{n}{j-1}$. Conversely, any projection operator onto any irreducible S_n submodule is an extremal element of the positive semidefinite cone.*

Problem: We have all the entries on the block diagonal for F. Find expressions in terms of Hahn polynomials for the remaining entries.

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