Some new constructions of imprimitive cometric association schemes

William J. Martin
Department of Mathematical Sciences
and
Department of Computer Science
Worcester Polytechnic Institute
Worcester, Massachusetts
martin@wpi.edu

Mikhail Muzychuk
Department of Computer Science and Mathematics
Netanya Academic College
Netanya 42365 Israel
muzy@netanya.ac.il

Jason Williford
Department of Mathematical Sciences
Worcester Polytechnic Institute
Worcester, Massachusetts
jsw@wpi.edu

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Dedicated to Professor Eiichi Bannai on the occasion of his 60$^{\text{th}}$ birthday.

Abstract

In a recent paper [9], the authors introduced the extended Q-bipartite double of an almost dual bipartite cometric association scheme. Since the association schemes arising from linked systems of symmetric designs are almost dual bipartite, this gives rise to a new infinite family of 4-class cometric schemes which are both Q-bipartite and Q-antipodal. These schemes, the schemes arising from linked systems, and all Q-polynomial bipartite distance-regular graphs enjoy a curious property: the relations restricted to any subcollection of the Q-antipodal classes is again a cometric association

scheme. We proved in [9] that this always holds for Q-antipodal schemes; we call such schemes "dismantlable". In this note, we explore a few examples of this phenomenon in more detail.

0.1 Outline of the paper

This paper has three meager goals. Mainly, we give an overview of the paper [9] by the same authors on which the talk in Sendai was based. In the process of doing this, we give a brief but self-contained account of linked systems of symmetric designs and a new family of 4-class Q-polynomial schemes based on them. Another goal is to discuss some specific examples of association schemes related to the results of [9].

1 Background material

Let $(X, \mathbf{A})$ be a symmetric d-class association scheme [1, 2] on v vertices with Schur idempotents $\mathbf{A} = \{A_0, \ldots, A_d\}$, primitive (ordinary) idempotents $E_0, \ldots, E_d$, eigenmatrices P and $Q = vP^{-1}$, valencies k_i and multiplicities m_j. When we say $(X, \mathbf{A})$ is *cometric* (or *Q-polynomial*), we imply that the ordering $E_0, E_1, \ldots, E_d$ is a Q-polynomial ordering. That is, the *Krein parameters* q_{ij}^k given by

$$E_i \circ E_j = \frac{1}{v} \sum_{k=0}^{d} q_{ij}^k E_k$$

(where $\circ$ denotes Schur — or entrywise — multiplication) satisfy

- $q_{ij}^k = 0$ whenever $k < |i-j|$ or $k > i+j$, and
- $q_{ij}^{i+j} > 0$ whenever $i + j \leq d$.

The parameters of a cometric scheme $(X, \mathbf{A})$ are entirely determined by its *Krein array*

$$\iota^*(X, \mathbf{A}) = \{b_0^*, b_1^*, \ldots, b_{d-1}^*; c_1^*, c_2^*, \ldots, c_d^*\}$$

where $b_j^* = q_{1,j+1}^j$, $c_j^* = q_{1,j-1}^j$ and we also define $a_j^* = q_{1j}^j = m_1 - b_j^* - c_j^*$.

A cometric scheme is *Q-bipartite* if all $a_j^* = 0$. This is equivalent to the condition that $q_{ij}^k = 0$ whenever $i + j + k$ is odd. A cometric scheme is *Q-antipodal* if $b_j^* = c_{d-j}^*$ for all j except possibly $j = \lfloor \frac{d}{2} \rfloor$.

An association scheme is *imprimitive* if some graph in the scheme is disconnected. As is well-known, it follows in this case that some union of relations is a non-trivial equivalence relation on X. In the next theorem and henceforth, we will let I_w (or simply I) denote the $w \times w$ identity matrix and J_r (or J) will denote the all ones matrix of order $r \times r$.

Theorem 1.1 (see [1, 2]). *The following are equivalent:*

(i) $(X, \mathbf{A})$ *is imprimitive;*

(ii) *for some* $j > 0$*,* E_j *has repeated columns;*

(iii) *for some subset* $\mathcal{I} = \{i_0 = 0, i_1, \ldots, i_s\}$ *of* $\{0, 1, \ldots, d\}$ *and some ordering of the vertices* $\sum_{h=0}^{s} A_{i_h} = I_w \otimes J_r$ *for some integers* w *and* r *with* $v = wr$*,* $1 < w, r < v$*;*

(iv) *for some subset* $\mathcal{J} = \{j_0 = 0, j_1, \ldots, j_s\}$ *of* $\{0, 1, \ldots, d\}$ *and some ordering of the vertices* $\sum_{h=0}^{s} E_{j_h}$ *has form* $\frac{1}{r}(I_w \otimes J_r)$ *for some integers* w *and* r *with* $v = wr$*,* $1 < w, r < v$*.*

(v) *The Bose-Mesner algebra* $\mathcal{A}$ *of the scheme contains a proper Schur-closed subalgebra of dimension at least two not containing* I*.*

The same scheme may have several such imprimitivity systems and our language must distinguish them; for example, the vertex set of a scheme which is Q-antipodal admits a partition into "Q-antipodal classes" and a scheme which is Q-bipartite has a "Q-bipartite imprimitivity system" which partitions the vertices into "dual bipartite classes". (If we are to further develop a theory of cometric schemes which are not distance-regular graphs, there is probably a need for more appropriate terminology.) In each case, we will use r for the size of a class and $w = v/r$ for the number of such classes in this partition. As above, the set $\mathcal{I}$ contains all indices $0 \le i \le d$ for which A_i has all zeros on blocks indexed by distinct classes and the set $\mathcal{J}$ contains all indices $0 \le j \le d$ for which E_j has all columns indexed by any class identical.

Theorem 1.2 (Suzuki [12]). *Suppose* $(X,\mathbf{A})$ *is an imprimitive cometric association scheme. Then one of the following holds:*

- $(X, \mathbf{A})$ *is* Q*-bipartite and* $\mathcal{J} = \{0, 2, 4, \ldots\}$*;*
- $(X, \mathbf{A})$ *is* Q*-antipodal and* $\mathcal{J} = \{0, d\}$*;*
- $d = 4$*,* $\iota^*(X, \mathbf{A}) = \{m, m-1, 1, b_3^*; 1, c_2^*, m - b_3^*, 1\}$ *and* $\mathcal{J} = \{0, 3\}$*;*
- $d = 6$*,* $\iota^*(X, \mathbf{A}) = \{m, m-1, 1, b_3^*, b_4^*, 1; 1, c_2^*, m - b_3^*, 1, c_5^*, m\}$ *(where* $a_2^* = a_4^* + a_5^*$*) and* $\mathcal{J} = \{0, 3, 6\}$*.* □

At this Sendai conference, Cerzo and Suzuki [6] showed that there are no association schemes of the third type in the list. No examples are known of the last type, either.

If $(X, \mathbf{A})$ is cometric with Q-polynomial ordering $0, 1, \ldots, d$, then the entries in column 1 of the matrix Q are all distinct and we may define a *natural ordering* on the relations by the requirement that $Q_{01} > Q_{11} > \cdots > Q_{d1}$. We will use this throughout the paper.

2 Overview of results

In this section, we give the results of [9] without proofs.

2.1 The extended Q-bipartite double

Our first result is a construction dual to the "extended bipartite double" construction of [2, Sec. 1.11].

We begin with the bipartite double. If we take any scheme with associate matrices A_i and primitive idempotents E_j ($0 \le i, j \le d$), then the bipartite double has associate matrices

$$A_i^+ = \begin{bmatrix} A_i & 0 \\ 0 & A_i \end{bmatrix} \qquad \text{and} \qquad A_i^- = \begin{bmatrix} 0 & A_i \\ A_i & 0 \end{bmatrix}$$

and primitive idempotents

$$E_j^+ = \frac{1}{2}\begin{bmatrix} E_j & E_j \\ E_j & E_j \end{bmatrix} \qquad \text{and} \qquad E_j^- = \frac{1}{2}\begin{bmatrix} E_j & -E_j \\ -E_j & E_j \end{bmatrix}.$$

A cometric scheme $(X, \mathbf{A})$ is *almost dual bipartite* if $a_j^* = 0$ for $j < d$ but $a_d^* \neq 0$. Bannai and Ito [1, p315] proved that the bipartite double of an almost dual bipartite cometric scheme is cometric as well, with Q-polynomial ordering $E_0^+, E_1^-, E_2^+, E_3^-, \ldots, E_0^-$. The Hermitian forms dual polar space graphs $[{}^2A_{2d-1}(r)]$ give an infinite family of examples, with arbitrarily large diameter, where this bipartite double is cometric but not metric. The scheme we now describe in Theorem 2.1(i) is called the *extended Q-bipartite double*.

Theorem 2.1. *Let $(X, \mathbf{A})$ be a d-class cometric association scheme on v vertices with primitive idempotents E_j and Krein parameters a_j^*, b_j^*, c_j^* satisfying $b_j^* + c_{j+1}^* = m_1 + 1$ for $0 \le j < d$. Then*

(i) *there exists an association scheme $(\hat{X}, \hat{\mathbf{A}})$ on $2v$ vertices where $\hat{X} = X \times \{0, 1\}$ and*

$$\hat{\mathbf{A}} = \left\{A_0^+,\ A_1^+ + A_d^-,\ A_2^+ + A_{d-1}^-,\ \ldots,\ A_0^-\right\}$$

where $A_0, A_1, A_2, \ldots$ is the natural ordering of Schur idempotents in the original scheme. Moreover, this is a Q-bipartite cometric scheme with Q-polynomial ordering

$$E_0^+,\ E_0^- + E_1^-,\ E_1^+ + E_2^+,\ E_2^- + E_3^-,\ \ldots,\ E_d^{\pm}$$

where the last matrix is E_d^+ if d is odd and E_d^- if d is even;

(ii) *the idempotent $E_1 + E_2$ generates a cometric fusion scheme $(X, \bar{\mathbf{A}})$ of the original scheme $(X, \mathbf{A})$; this is the Q-bipartite quotient of the scheme $(\hat{X}, \hat{\mathbf{A}})$.*

The Krein array for the fusion scheme in part (ii) is

$$\iota^*(X, \hat{\mathbf{A}}) = \left\{m_1 + m_2, \frac{b_1^* b_2^*}{c_2^*}, \frac{b_3^* b_4^*}{c_2^*}, \ldots, \frac{b_{d-2}^* b_{d-1}^*}{c_2^*};\ 1, \frac{c_3^* c_4^*}{c_2^*}, \frac{c_5^* c_6^*}{c_2^*}, \ldots, \frac{c_d^*(m_1 + 1)}{c_2^*}\right\}$$

when d is odd and

$$\iota^*(X, \hat{\mathbf{A}}) = \left\{m_1 + m_2, \frac{b_1^* b_2^*}{c_2^*}, \frac{b_3^* b_4^*}{c_2^*}, \ldots, \frac{b_{d-3}^* b_{d-2}^*}{c_2^*};\ 1, \frac{c_3^* c_4^*}{c_2^*}, \frac{c_5^* c_6^*}{c_2^*}, \ldots, \frac{c_{d-1}^* c_d^*}{c_2^*}\right\}$$

when d is even.

Example 2.2 (A. Munemasa, personal communication). *The Soicher graph* Σ *for* $M_{22} : 2$ *is a distance-regular graph of diameter three having intersection array* $\iota(\Sigma) = \{110, 81, 12; 1, 18, 90\}$. *The underlying association scheme* $(X, \mathbf{A})$ *is cometric with Krein array* $\iota^*(X, \mathbf{A}) = \{55, 49, 21; 1, 7, 35\}$ *so Theorem 2.1(i) applies. We then obtain a 4-class scheme which is both metric and cometric. This distance-regular graph was discovered by Meixner and has intersection array* $\{176, 135, 24, 1; 1, 24, 135, 176\}$. *If* A_0, A_1, A_2, A_3 *are the distance matrices of the Soicher graph, then Munemasa observed that the adjacency matrix of the Meixner graph can be expressed as*

$$A_1^+ + A_3^- = \begin{bmatrix} A_1 & A_3 \\ A_3 & A_1 \end{bmatrix}$$

as in part **(i)** *of the theorem.* □

Several open parameter sets for diameter three cometric distance-regular graphs also satisfy the conditions of Theorem 2.1. These include

$$\begin{array}{lr} v = 322 & \{60, 45, 8;\ 1, 12, 50\} \\ v = 392 & \{69, 56, 10;\ 1, 14, 60\} \\ v = 378 & \{78, 50, 9;\ 1, 15, 60\} \\ v = 800 & \{119, 100, 15;\ 1, 20, 105\} \\ v = 900 & \{174, 110, 18;\ 1, 30, 132\}. \end{array}$$

Example 2.3. *The block scheme of the* 4-$(11, 5, 1)$ *Witt design is a cometric scheme with Krein array* $\iota^*(X, \mathbf{A}) = \{10, \frac{242}{27}, \frac{11}{5}; 1, \frac{55}{27}, \frac{44}{5}\}$. *Clearly the conditions of Theorem 2.1(i) are met. But the extended Q-bipartite double of this scheme is already well-known: it is the block scheme of the* 5-$(12, 6, 1)$ *Witt design with Krein array* $\iota^*(X,\mathbf{A}) = \{11, 10, \frac{242}{27}, \frac{11}{5}; 1, \frac{55}{27}, \frac{44}{5}, 11\}$. *By the same token, the induced association scheme on the even subcode of the perfect binary Golay code (i.e., the dual scheme of the coset graph of the perfect code), with Krein array* $\iota^*(X, \mathbf{A}) = \{23, 22, 21;\ 1, 2, 3\}$, *has as its extended Q-bipartite double the induced scheme on the extended binary Golay code with Krein array* $\iota^*(X, \mathbf{A}) = \{24, 23, 22, 21;\ 1, 2, 3, 24\}$.

2.2 Linked systems of symmetric designs

A *symmetric* (v, k, λ) *design* [8] is an incidence structure based on two disjoint sets $\mathcal{P}_1$ and $\mathcal{P}_2$ of v objects each, called *points* and *blocks*, respectively, such that

- each block is incident with k points;
- any two distinct blocks are incident with λ common points;
- each point is incident with k blocks;
- any two distinct points are incident with λ common blocks;
- the design is non-degenerate: $0 < \lambda < k < v$.

Clearly the definition is symmetric in the role of points and blocks. The incidence graph G of such a design is a bipartite distance-regular graph of diameter three with intersection array $\iota(G) = \{k, k-1, k-\lambda; 1, \lambda, k\}$. Such an incidence graph is always cometric and every bipartite distance-regular graph with diameter three arises in this way. If we order the eigenvalues of G in decreasing order,

$$k > \sqrt{n} > -\sqrt{n} > -k$$

where $n = k - \lambda$ is the *order* of the design, this yields a Q-polynomial ordering E_0, E_1, E_2, E_3 of the primitive idempotents for the Bose-Mesner algebra $\mathcal{A}$ of the scheme. (Note that E_0, E_2, E_1, E_3 is a second Q-polynomial ordering.) The rank of E_1 is $m_1 = v - 1$. This is a Q-antipodal 3-class scheme with Q-antipodal imprimitivity system $\{\mathcal{P}_1, \mathcal{P}_2\}$. For more information on symmetric designs, the reader is referred to Lander's monograph [8].

We now summarize some material from [4] and [10]. A *linked system of ℓ symmetric (v, k, λ) designs* is a graph G defined on a vertex set

$$X = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_{\ell+1}$$

where $\pi = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{\ell+1}\}$ is a partition of X into $\ell + 1$ sets of size v each, enjoying the following properties:

1. partition π is a proper coloring of G: no edge of G has both ends in the same class $\mathcal{P}_i$;

2. for any $i \neq j$, the subgraph of G induced on $\mathcal{P}_i \cup \mathcal{P}_j$ is the incidence graph of a symmetric (v, k, λ) design;

3. for any three distinct classes $\mathcal{P}_i, \mathcal{P}_j, \mathcal{P}_k$, the number of common neighbors of a vertex x in $\mathcal{P}_i$ and a vertex y in $\mathcal{P}_j$ which lie in $\mathcal{P}_k$ depends only on whether x and y are adjacent in G or not; it does not depend on the choice of x and y nor on the choice of i, j and k.

Let σ denote the number of common neighbors in $\mathcal{P}_k$ of x in $\mathcal{P}_i$ and y, adjacent to x, in $\mathcal{P}_j$ $(i \neq j \neq k \neq i)$. Let τ denote the same parameter for x and y non-adjacent in G. Then we have, by double-counting

$$\{(x, y, z)|\ x \sim z \sim y,\ y \in \mathcal{P}_j,\ z \in \mathcal{P}_k\}$$

for fixed $x \in \mathcal{P}_i$,

$$k\sigma + (v - k)\tau = k^2. \tag{2.1}$$

In determining σ and τ, Cameron [4] next considers the blocks M_{ij} of the adjacency matrix of G:

$$A_1 = \begin{bmatrix} 0 & M_{12} & M_{13} & \ldots \\ M_{21} & 0 & M_{23} & \ldots \\ M_{31} & M_{32} & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $M_{ji} = M_{ij}^T$ is the submatrix recording adjacencies in G from vertices in $\mathcal{P}_j$ to vertices in $\mathcal{P}_i$. Noting that

$$\det(M_{ij}) = \det(M_{ij}^T) = kn^{\frac{v-1}{2}}$$

for all $i \neq j$, he takes determinants of both sides of the equation

$$M_{ik}M_{jk}^T = (\sigma - \tau)M_{ij} + \tau J$$

to obtain

$$k^2 n^{v-1} = kn^{\frac{v-1}{2}}(\sigma - \tau)^{v-1}\frac{1}{k}(k\sigma + (v-k)\tau).$$

Using Equation (2.1) and simplifying, we find

$$(\tau - \sigma)^2 = n,$$

yielding

$$\sigma = \frac{1}{v}\left(k^2 \pm \sqrt{n}(v-k)\right), \qquad \tau = \frac{k}{v}\left(k \mp \sqrt{n}\right)$$

where signs are chosen appropriately to make σ and τ integers. It follows from $k^2 \equiv n \bmod v$ and $v > 2n$ that at most one choice of signs gives us integer values for σ and τ. Replacing the designs by their complements yields a complementary system of linked $(v, v-k, v-2k+\lambda)$-designs. Note that the complementary system has the same order n, but the signs in the corresponding formulae for its σ and τ are opposite to the ones of the original system. Thus we can always assume that

$$\sigma = \frac{1}{v}(k^2 - \sqrt{n}(v-k)), \ \ \tau = \frac{k}{v}(k + \sqrt{n}).$$

Note that this fixes $\tau - \sigma$ to be $\sqrt{n}$, and we can no longer assume $2k < v$ as is customary in design theory.

Assume now that G is a linked system of ℓ symmetric (v, k, λ) designs. Since σ is an integer, $\ell > 1$ implies that the order $n = k - \lambda$ is a perfect square. We then obtain a 3-class association scheme $(X, \mathbf{B})$ with associate matrices

$$B_0 = I, \quad B_1 = A(\tilde{G}), \quad B_2 = A(G_2), \quad B_3 = A(G)$$

where G_2 is the union of the $\ell + 1$ complete graphs on the Q-antipodal classes $\mathcal{P}_i$ and $\tilde{G}$ is the multipartite complement of G; this is a linked system of ℓ symmetric $(v, v-k, v-2k+\lambda)$ designs. These matrices satisfy

$$B_i B_j = \sum_{h=0}^{3} p_{ij}^h B_h$$

where the intersection numbers p_{ij}^h, first given by Mathon [10], are recorded in the matrices $L_i = [p_{ij}^h]_{h,j}$ below:

$$L_1 = \begin{bmatrix} 0 & \ell(v-k) & 0 & 0 \\ 1 & \frac{\ell-1}{v}\left[(v-k)^2 + k\sqrt{n}\right] & v-k-1 & (\ell-1)\frac{k}{v}\left[v-k-\sqrt{n}\right] \\ 0 & \ell(v-2k+\lambda) & 0 & \ell(k-\lambda) \\ 0 & (\ell-1)\frac{v-k}{v}\left[v-k-\sqrt{n}\right] & v-k & (\ell-1)\frac{v-k}{v}\left[k+\sqrt{n}\right] \end{bmatrix},$$

$$L_2 = \begin{bmatrix} 0 & 0 & v-1 & 0 \\ 0 & v-k-1 & 0 & k \\ 1 & 0 & v-2 & 0 \\ 0 & v-k & 0 & k-1 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 0 & 0 & 0 & \ell k \\ 0 & (\ell-1)\frac{k}{v}\left[v-k-\sqrt{n}\right] & k & (\ell-1)\frac{k}{v}\left[k+\sqrt{n}\right] \\ 0 & \ell(k-\lambda) & 0 & \ell\lambda \\ 1 & (\ell-1)\frac{v-k}{v}\left[k+\sqrt{n}\right] & k-1 & \frac{\ell-1}{v}\left[k^2-(v-k)\sqrt{n}\right] \end{bmatrix}.$$

(Trivially, $L_0 = I$.) The eigenmatrices for this scheme are then

$$P = \begin{bmatrix} 1 & \ell(v-k) & v-1 & \ell k \\ 1 & \ell\sqrt{n} & -1 & -\ell\sqrt{n} \\ 1 & -\sqrt{n} & -1 & \sqrt{n} \\ 1 & k-v & v-1 & -k \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & v-1 & \ell(v-1) & \ell \\ 1 & \frac{k}{\sqrt{n}} & -\frac{k}{\sqrt{n}} & -1 \\ 1 & -1 & -\ell & \ell \\ 1 & \frac{k-v}{\sqrt{n}} & \frac{v-k}{\sqrt{n}} & -1 \end{bmatrix}$$

and the Krein array is $\iota^*(X, \mathbf{B}) = \{b_0^*, b_1^*, b_2^*; c_1^*, c_2^*, c_3^*\}$ where

$$b_0^* = c_3^* = m = v-1, \qquad b_1^* = \ell c_2^* = \frac{\ell}{\ell+1}\left(v-2+\frac{1}{\sqrt{n}}(v-2k)\right), \qquad b_2^* = c_1^* = 1.$$

For $\ell > 1$, the scheme is no longer metric and there is only one Q-polynomial ordering of the primitive idempotents. In the matrices above, we have used the natural ordering of relations, ensuring $Q_{01} > Q_{11} > Q_{21} > Q_{31}$.

2.3 New family of 4-class Q-antipodal Q-bipartite schemes

Let $\mathbf{B} = \{B_0, B_1, B_2, B_3\}$ be the associate matrices of the association scheme arising from a linked system of symmetric designs on vertex set X with parameters

$$v = 16s^2, \qquad n = 4s^2, \qquad k = 2s(4s-1), \qquad \lambda = 2s(2s-1)$$

where s is a positive integer. (From above, for $\ell > 1$, we need the order n to be a square. We will need $v = 2k + 2\sqrt{n}$ in order to ensure $b_1^* + c_2^* = m_1 + 1$. So our design must satisfy $v = 4n$ and n must be even for p_{11}^1 to be integral when ℓ is even.) Now applying Theorem 2.1, we obtain the following:

Theorem 2.4. *Let $(X, \mathbf{B})$ be a linked system of symmetric designs with parameters $v = 16s^2$, $k = 2s(4s-1)$ and $\lambda = 2s(2s-1)$ and let $Y = X \times \{0,1\}$. Consider $\mathbf{A} = \{A_0 = I, A_1, A_2, A_3, A_4\}$ given by*

$$A_1 = \begin{bmatrix} B_3 & B_1 \\ B_1 & B_3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} B_2 & B_2 \\ B_2 & B_2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} B_1 & B_3 \\ B_3 & B_1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Then $(Y, \mathbf{A})$ is a 4-class Q-antipodal Q-bipartite cometric association scheme.

The Krein array for this scheme is $\iota^*(Y,\mathbf{A}) = \{v, v-1, \ell\frac{v}{\ell+1}, 1;\ 1, \frac{v}{\ell+1}, v-1, v\}$ which shows that any such scheme is both Q-bipartite and Q-antipodal.

Now we demonstrate that there are linked systems of designs known which satisfy the conditions of the above theorem.

In fact, only one infinite family of linked systems of symmetric designs is known [4]. A description of these linked systems based on the Kerdock codes is cited in [4] as private communication from J. M. Goethals. The first published description is based on the "Cameron-Seidel scheme" [5].

The Cameron-Seidel scheme corresponds to a system of $\ell = 2^{2t+1} - 1$ linked symmetric $(2^{2t+2}, 2^{2t+1} - 2^t, 2^{2t} - 2^t)$ designs, where t can be any positive integer. So, by deleting Q-antipodal classes, we obtain a system of ℓ such linked designs for any $\ell < 2^{2t+1}$. Each of these systems has order $n = k - \lambda = 2^{2t}$ an even square and $v = 4n$. So the construction given in Theorem 2.4 applies and we have an infinite family of 4-class Q-antipodal Q-bipartite cometric association schemes with $s = 2^{t-1}$ in the language above.

We note that the Krein parameters are easily computed from the second eigenmatrix Q:

$$Q = \begin{bmatrix} 1 & v & (\ell+1)(v-1) & \ell v & \ell \\ 1 & 2^{t+1} & 0 & -2^{t+1} & -1 \\ 1 & 0 & -\ell-1 & 0 & \ell \\ 1 & -2^{t+1} & 0 & 2^{t+1} & -1 \\ 1 & -v & (\ell+1)(v-1) & -\ell v & \ell \end{bmatrix}.$$

The Krein array is given above. An important feature to note is that, unless $\ell + 1$ divides v, the Krein parameter $c_2^* = q_{11}^2$ is non-integral. So these schemes cannot be duals of metric schemes in general. Moreover, for $\ell > 1$, the schemes cannot be metric since they are Q-antipodal with more than two Q-antipodal classes.

2.4 Structure of Q-bipartite schemes

A bipartite distance-regular graph obviously has $w = 2$ bipartite halves; the next theorem is dual to this.

Theorem 2.5 ([3]). *If $(X, \mathbf{A})$ is Q-bipartite with w dual bipartite classes of size r each, then $r = 2$.*

A special case of this result has been known for some time: a Q-polynomial antipodal distance-regular graph must be a double cover of its folded graph [2, Theorem 8.2.4].

Still assuming the natural ordering on relations, in the Q-bipartite case with vertices ordered so that dual bipartite pairs appear consecutively, this gives $A_0 + A_d = I_{v/2} \otimes J_2$. Moreover, the dual bipartite classes are mapped via $u_1 : v \mapsto E_1 v$ to opposing points on lines through the origin in V_1 (or in $\mathbb{R}^m$). This observation gives us the following two corollaries.

Corollary 2.6 ([9]). *Let $(X, \mathbf{A})$ be a Q-bipartite cometric association scheme with the natural ordering on relations. Then, for the first eigenspace, the sequence $m_1 = Q_{01} > Q_{11} > \cdots > Q_{d1}$ is symmetric about the origin. In particular, $Q_{\frac{d}{2},1} = 0$ whenever d is even.*

Corollary 2.7 ([9]). *Let* $(X, \mathbf{A})$ *be a* Q*-bipartite cometric association scheme with natural ordering on relations. Then the intersection numbers satisfy*

$$p_{ij}^{k} = p_{i,d-j}^{d-k}$$

for all $0 \leq i, j, k \leq d$*.*

2.5 Structure of Q-antipodal schemes

Gardiner proved (cf. [2, Prop. 4.2.2]): for any antipodal distance-regular graph of valency k, the index r of the cover is bounded above by k. Therefore, we expect the number, w, of Q-antipodal classes to be bounded above by the first multiplicity, m_1. The following result is a bit weaker.

Theorem 2.8 ([9]). *Let* $(X, \mathbf{A})$ *be a* d*-class* Q*-antipodal association scheme with* w Q*-antipodal classes of size* r *each. If* d *is odd, then* $w \leq m_1$*. If* d *is even, then* $w \leq m_2$*.*

Considering sign change patterns in columns of the matrix Q, we use some results on Sturm sequences to obtain the following:

Corollary 2.9 ([9]). *In any* d*-class* Q*-antipodal scheme,* $\lfloor \frac{d}{2} \rfloor$ *non-trivial relations occur between vertices in the same* Q*-antipodal class and* $\lceil \frac{d}{2} \rceil$ *relations occur between classes. Namely, for* i *odd, the partition into* Q*-antipodal classes is a proper vertex coloring of graph* G_i *and for* i *even, each component of* G_i *lies entirely within some* Q*-antipodal class.*

The intersection numbers of Q-antipodal schemes behave in a manner similar to those of bipartite graphs.

Corollary 2.10 ([9]). *Let* $(X, \mathbf{A})$ *be a* Q*-antipodal cometric association scheme with relations ordered naturally. Then the intersection numbers satisfy*

$$p_{ij}^{k} = 0$$

unless either $i + j + k$ *is even or* ijk *is odd.*

A Q-antipodal cometric association scheme with Q-antipodal classes $X_1, \ldots, X_w$ is *dismantlable* if, for any proper subset $\{X_{i_1}, \ldots, X_{i_{w'}}\}$ of its Q-antipodal classes the set of induced graphs $(G_i)_{Y \times Y}$ where $0 \leq i \leq d$ and $Y = X_{i_1} \cup \cdots \cup X_{i_{w'}}$ is again an association scheme. For $w' = 1$, it is a standard result (originally due to Rao, Ray-Chaudhuri and Singhi – see [2, Section 2.4]) that we find a subscheme on each X_i and these all have the same parameters. (Here, we call this the *local scheme*.)

Theorem 2.11. *Every* Q*-antipodal scheme is dismantlable. The subscheme induced on any non-trivial collection of* w' Q*-antipodal classes is cometric for* $w' \geq 1$ *and* Q*-antipodal with* d *classes for* $w' > 1$*.*

Example 2.12. *The first example in the family constructed in Theorem 2.4 has 96 vertices. Here, we study a smaller example, which is somewhat degenerate. We begin with a 3-class scheme on 12 vertices which comes from the symmetric* $(4, 3, 2)$ *design (3-cube). (While, under an appropriate ordering of idempotents,* $k > i + j$ *implies* $q_{ij}^k = 0$*, this scheme is not cometric). Our 4-class scheme has twenty-four vertices and* $m_1 = 4$.

While the positioning of one bipartite half of the vertices of a regular 3-cube in $\mathbb{R}^3$ *uniquely determines the location of the remaining four vertices, for the 4-cube in* $\mathbb{R}^4$*, there are two solutions for the placement of the "black vertices" given feasible coordinates for all the white vertices. What is more, these two sets of 8 black vertices themselves form a regular Euclidean 4-cube. What we have just described is the geometry of the first eigenspace of our association scheme. The twenty-four points so described in* $\mathbb{R}^4$ *are the vertices of the well-known 24-cell. Since the polytope is antipodal, the association scheme is also Q-bipartite.*

The Q-antipodal classes for this scheme are the three 8-sets described above, each of which has a 16-cell (or hyperoctahedron) as its convex hull. The Q-antipodal property is reflected in the relative positioning of these three 8-vertex polyhedra. Between any two Q-antipodal classes, the induced subgraph of the graph corresponding to the matrix A_1 *in our scheme is the graph of the 4-cube* $H(4, 2)$*. It is easy to check that there is no way to pack a fourth 16-cell into this configuration and retain the pairwise relationships we have described. We believe that this Euclidean packing problem is at the heart of the study of Q-antipodal cometric schemes.* □

Example 2.13. *The coset graph of the shortened ternary Golay code, labeled (A17) in [2, p365] has intersection array* $\{20, 18, 4, 1; 1, 2, 18, 20\}$*; this is an antipodal distance-regular graph belonging to a translation scheme. The dual association scheme is Q-antipodal on* $v = 243$ *vertices with* $w = 3$ *Q-antipodal classes. Removing one of these, we obtain a Q-antipodal scheme on 162 vertices having* $w = 2$ *Q-antipodal classes which is not metric. Note that this scheme has parameters*

$$d = 4,\ v = 162,\ \iota^*(X, \mathbf{A}) = \{20, 18, 3, 1; 1, 3, 18, 20\}$$

formally dual to those of an unknown diameter four bipartite distance-regular graph, but it is not realizable as a translation scheme.

Example 2.14. *The same idea applied to graphs labeled (A16) and (A18) on page 365 of [2] yield new Q-antipodal schemes with parameters*

$$d = 5,\ v = 486,\ \iota^*(X, \mathbf{A}) = \{22, 20, \frac{27}{2}, 2, 1;\ 1, 2, \frac{27}{2}, 20, 22\},\ w = 2$$

$$d = 6,\ v = 1536,\ \iota^*(X, \mathbf{A}) = \{21, 20, 16, 8, 2, 1;\ 1, 2, 4, 16, 20, 21\},\ w = 3.$$

On the same page of [2], the dual of this last scheme is ruled out by counting hexagons.

We briefly mention some more special cases of Theorem 2.11. A Q-polynomial distance-regular graph is Q-antipodal if and only if it is bipartite. The Q-antipodal classes are precisely the $w = 2$ bipartite color classes. The induced configuration on either one of these classes has long been known to be a cometric association scheme (see [2, Prop. 4.2.2] and preceding discussion). So the theorem is trivial in the metric case. It clearly also holds (by definition) for any linked system of symmetric designs and therefore also for the new family of 4-class schemes introduced in Section 2.2.

The octahedron is a Q-antipodal scheme with $r = 2$ and $w = 3$. In this case, $m_1 = 3$ while $m_1 = 2$ for any induced subscheme on $w' = 2$ Q-antipodal classes. In spite of this, for Q-antipodal schemes with three or more classes, these two multiplicities coincide.

Theorem 2.15 ([9]). *For a Q-antipodal d-class association scheme with $d \geq 3$ and w Q-antipodal classes of size r, the first multiplicity m_1 does not depend on w but only on the parameters of the local scheme.*

Indeed, bounds much better than that given in Theorem 2.8 are known in the case of linked systems of symmetric designs (i.e., $d = 3$). For example, we have the trivial bound $w \leq 2$ when n is not a square. Moreover, since $m_2 = (w-1)m_1$ in this case, we obtain

$$1 + m_2 = 1 + (w-1)m_1 \leq \mathrm{rank}(E_1 \circ E_1) \leq \frac{1}{2}m_1(m_1+1)$$

giving $w \leq \frac{m+2}{2}$. Mathon [10] and Noda [11] give stronger bounds for the number of linked symmetric (v, k, λ) designs, but only in the case when the quantity $(\sigma - \tau)(v - 2k)$ is positive. For example, we note that their bounds do not apply to the case $v = 36$, $n = 9$.

In the paper [9], we give a list of all cometric association schemes known to us which are not metric. Another version of this list is maintained on the web at
`http://users.wpi.edu/~martin/RESEARCH/QPOL/`.

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