CIMPA Philippines
Semidefinite Prog. in Alg. Combin.
July 28, 2009

Student Problem Set 4

Instructions: Please solve these problems in today's problem session at 4:30 (and continue later, as needed) in groups and present the solutions on the black/whiteboards in the front of the room. You are encouraged to ask questions and to ask for help.

1. A subset A of a Euclidean space E is *convex* if, for all $x, y \in A$ and all $t \in [0, 1]$, $tx + (1 - t)y$ belongs to A as well. Prove that every cone in a Eulcidean space is convex.

2. Prove that the Hadamard (or Schur) product $X \circ Y$ of two psd matrices X and Y (of the same size) is also psd. *[HINT: Use the spectral decomposition.]*

3. A subset A of a Euclidean space E is an *affine subspace* if, for all $x, y \in A$ and all $t \in \mathbb{R}$, $tx + (1 - t)y$ belongs to A. Prove that the solution set to any linear system $\{a_i \cdot x = b_i : 1 \leq i \leq m\}$ is an affine subspace.

4. Finish Frank's proof from Friday that every non-negative polynomial over the reals is expressible as a sum of squares of polynomials. That is, if

$$f(x) = \prod_{i=1}^{r} (x - \beta_i)(x - \bar{\beta}_i)$$

for some complex nubmers β_i, then $f = \sum_\ell g_\ell^2$ for some real polynomials g_ℓ. [HINT: *Just show that*

$$(x - \beta)(x - \bar{\beta}) = (x - \mathcal{R}e\beta)^2 + (\mathcal{I}m\beta)^2.]$$

(Extra credit: For a, b, c, d, verify

$$(a^2 + b^2)(c^2 + d^2) = (ad - bc)^2 + (bd + ac)^2$$

to show that p is expressible as a sum of only two squares.)

5. For a given polynomial $p(x)$, express the following optimization problem in standard SDP form:

$$\max\ \alpha$$

subject to

$$p(x) - \alpha = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^d \end{pmatrix}^t Q \begin{pmatrix} 1 \\ x \\ \vdots \\ x^d \end{pmatrix}$$

where Q is a psd matrix of order $d + 1$.

6. Solve the above SDP by hand for $p(x) = ax^2 + bx + c$, $a > 0$.

7. If $p(x)$ is a SOS of degree $2d$, find a psd matrix Q of order $d+1$ such that

$$p(x) = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^d \end{pmatrix}^t Q \begin{pmatrix} 1 \\ x \\ \vdots \\ x^d \end{pmatrix}.$$

8. Let $\mathcal{K} = S^n_{\geq 0}$ be the cone of positive semidefinite matrices in the Euclidean space $E = S^n$ of all symmetric $n \times n$ matrices with inner product $M \cdot N = \text{trace}(MN)$. Frank proved that $\mathcal{K}$ contains its dual cone $\mathcal{K}^*$. Now prove that $\mathcal{K} \subseteq \mathcal{K}^*$.

9. Find a simple example of a cone which is not equal to its dual. *[HINT: Take the complete space E or any linear subspace of $\mathbb{R}^n$.]*

10. Finish Frank's proof of the Duality Theorem.

11. Let W' be a symmetric $n \times n$ matrix with non-negative entries and zero diagonal. Define the $n \times n$ matrix W by

$$W_{ij} = \begin{cases} \sum_{k=1}^n W'_{ik}, & \text{if } i = j; \\ -W'_{ij}, & \text{if } i \neq j. \end{cases}$$

Prove that $W \succeq 0$.