CIMPA Philippines
Semidefinite Prog. in Alg. Combin.
July 21, 2009

Student Problem Set 2

Instructions: Please solve these problems in today's problem session at 4:30 (and continue later, as needed) in groups and present the solutions on the black/whiteboards in the front of the room. You are encouraged to ask questions and to ask for help.

1. If a and b are vertices of the n-cube and $\mathrm{dist}(a,b)=i$, compute
$$\sum_{\mathrm{wt}(c)=j} (-1)^{c\cdot a}(-1)^{c\cdot b}.$$
[HINT: The answer is $\sum_{\ell=0}^{j}(-1)^\ell\binom{i}{\ell}\binom{n-i}{j-\ell}$*.]*

2.

3. We find generators for the *automorphism group* of the n-cube Q_n:
   (a) Prove that, for any $c\in\mathbb{Z}_2^n$, the permutation $\phi_c : X\to X$ which sends a to $a+c$ preserves adjacency (i.e., for any $a,b\in X$, a is adjacent to b if and only $a+c$ is adjacent to $b+c$;
   (b) Prove that, for any $\tau\in S_n$ (the symmetric group on n letters), the permutation $\hat{\tau} : X\to X$ which sends $a=(a_1,\ldots,a_n)$ to $(a_{\tau(1)},\ldots,a_{\tau(n)})$ preserves adjacency.

4. Prove $A_iA_j=\sum_{k=0}^{n} p_{ij}^k A_k$ where $p_{ij}^k=\binom{k}{\ell}\binom{n-k}{i-\ell}$ where $\ell := (i+k-j)/2$.

5. Prove that, for $i\neq j$, $\sum_{\ell=0}^{n}\binom{n}{\ell}K_i(\ell)K_j(\ell)=0$. What is this sum when $i=j$? *[HINT: Use the orthogonality of characters and the expression where we first saw* $K_i(t)$*.]*

6. Show that A_0 is the identity matrix.

7. Show that $A_0+A_1+\cdots+A_n=J$, the all-ones matrix.

8. Show that A_i has constant row sum $\binom{n}{i}$.

9. Prove that $A_1A_i=(n-i+1)A_{i-1}+(i+1)A_{i+1}$ and use this, together with a simple induction argument, to prove that A_i is expressible as a polynomial of degree i in A_1.

10. Prove that the vector space $\mathbb{A}$ is closed under entrywise multiplication of matrices: if $M=[m_{ij}]$ and $N=[n_{ij}]$, then $M\circ N$ is the matrix with (i,j)-entry $m_{ij}n_{ij}$.

11. Prove that each A_i and the sum of any subset of $\{A_0,\ldots,A_n\}$ is idempotent under $\circ$.

12. Give a full description of the positive semidefinite cone in the space of symmetric 2×2 matrices.

13. Prove that every linear programming problem can be expressed in any one of the following three forms:

$$\begin{array}{ccc} \max c^\top x & \max c^\top x & \max c^\top x \\ Ax \leq b & Ax = b & Ax \leq b \\ x \geq 0 & x \geq 0 & \end{array}$$

by changing the number of variables and/or constraints.

14. Write down the linear programming problem for a binary code of length 7 and minimum distance $\delta \geq 3$. With the added assumption that $a_7 = 1$ (and a bit of thinking), solve this LP to find the optimal weight distribution of such a code.

15. For $n = 3$, consider the space of real-valued polynomials in x, y, z. Find a harmonic polynomial that is equal to the (not harmonic) polynomial x^2 on the entire sphere.

16. **(a)** Determine the dimension of $\mathcal{P}_k$, the vector space of polynomials on $\mathbb{R}^n$ of degree at most k.
(b) How show that the space $\mathcal{H}_k$ of harmonic polynomials of degree at most k on $\mathbb{R}^n$, has dimension

$$\binom{n+k-1}{k} + \binom{n+k-2}{k-1}.$$

17. For $n = 2$, the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is expressed in polar coordinates (r, θ) as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

18. if f and g are functions on $\mathbb{R}^n$ and all the following integrals make sense, then

$$\int_{\mathbb{R}^n} f(\Delta g) = \int_{\mathbb{R}^n} (\Delta f) g.$$