CIMPA Philippines
Semidefinite Prog. in Alg. Combin.
July 20, 2009

Student Problem Set 1

Instructions: Please solve these problems now (and later) in groups and present the solutions on the black/whiteboards in the front of the room. You are encouraged to ask questions and to ask for help.

1. Under the binary symmetric channel, what is the probability that the received vector y is within distance t of the transmitted codeword c?

2. For $n = 4$ and $\delta = 1, 2, 3, 4$, find a code C of maximum size in H_4 with $d(C) \geq \delta$.

3. Explain why a code C with $d(C) \geq \delta$ can be used to **detect** any error of up to $\delta - 1$ coordinate positions in the transmitted vector.

4. Make a table of the values $A(n, \delta)$ for $1 \leq n \leq 4$ and $1 \leq \delta \leq n$.

5. The *binary repetition code* of length n is the dual of the "parity check" code PC. Find all of its codewords.

6. Find all parameters $[n, k, \delta]$ and the error-correction radius r for the binary repetition code.

7. Prove that a binary linear code C has minimum distance at least three if and only if its parity check matrix H has no all-zero columns and no repeated columns.

8. Find a simple geometric condition on the columns of the parity check matrix H under which the linear code C has minimum distance at least four.

9. Find a generator matrix for the extended Hamming code $\mathcal{H}^{ext}$ coming from the $[7, 4, 3]$ Hamming code.

10. Find a parity check matrix for this same code.

11. Prove that $\mathcal{H}^{ext}$ is an $[8, 4, 4]$-code.

12. Prove that $\mathcal{H}^{ext}$ is a self-dual code: it is equal to its dual code.

13. A code C is called a *single error-correcting code* if it has minimum distance three or four. Prove that a binary single error-correcting code C achieving the Hamming bound must have length $n = 2^r - 1$ for some integer r.

14. For which $n \leq 10$ can a perfect binary double error-correcting code exist?

15. Use Professor Bachoc's ideas to compute the full weight distribution of the perfect binary Golay code, a $[23, 12, 7]$-code.

16. Use the Plotkin bound and a simple construction to exactly compute $A(n, \delta)$ for $\delta = \frac{2}{3}n$ for all n divisible by three.

17. Use the Plotkin bound to compute $A(n, \delta)$ for $\delta > \frac{3}{4}n$ for all $n \geq 2$.

18. Use the MacWilliams identities to decide if any of the following binary linear code of given length n exist with weight distribution given as follows:

    (a) $n = 6, \quad A_0 = 1, \quad A_1 = A_2 = 0, \quad A_3 = 4, \quad A_4 = 3, \quad A_5 = A_6 = 0.$

    (b) $n = 5, \quad A_0 = 1, \quad A_1 = A_2 = 0, \quad A_3 = 2, \quad A_4 = 1, \quad A_5 = 0.$

    (c) $n = 5, \quad A_0 = 1, \quad A_1 = A_2 = 0, \quad A_3 = 1, \quad A_4 = 2, \quad A_5 = 0.$

19. In the orthogonal group $O(2)$, let T be the matrix which rotates a vector about the origin by an angle $\psi = \pi/3$. Find the matrix representing T with respect to the standard basis.

20. In the orthogonal group $O(2)$, let R be the matrix which reflects a vector across the line $y = \frac{1}{\sqrt{3}}x$. Find the matrix representing R with respect to the standard basis.

21. Now find the matrices representing the product transformations TR and RT in $O(2)$ and describe these two transformations geometrically.

22. Explain why the vector space P_1 (in two variables) is preserved by $O(2)$. Show exactly how a polynomial
$$f = a + bx + cy$$
is transformed by the orthogonal matrix $A = [a_{ij}]$. In your solution, did you make use of the fact that A satisfies $AA^t = I$?

23. Use the Laplace operator for $n = 2$ to find bases for the harmonic spaces V_k for $k = 2, 4, 5$.

24. Use the Laplace operator
$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$
for $n = 3$ to find bases for the spaces of harmonic polynomials in x, y, z of degree $k = 1, 2, 3$.

25. Calculate the inner product of $f(x, y) = x^2 - y^2$ and $g(x, y) = 2xy$ in $L^2(S^1)$.

26. Consider the perfect binary Hamming code $\mathcal{H}$ of length $n = 7$. Show that the codewords of any given weight k form a 2-$(7, k, \lambda)$ design. Compute λ for each $k = 0, 1, \ldots, 7$.

27. Prove that $X \subset S^{n-1}$ is a spherical 1-design if and only if the vectors in X sum to the origin in $\mathbb{R}^n$.

28. Compute the kissing number $k(n)$ in dimensions $n = 1, 2$ and say as much as you can about $k(3)$ and $k(4)$.