Hermitian adjacency matrix of digraphs

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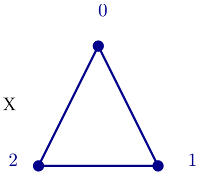

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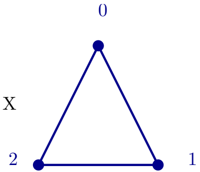
$$Spec(X) = \{2, (-1)^{(2)}\}$$



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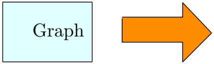

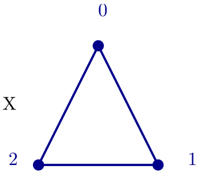

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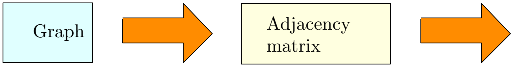

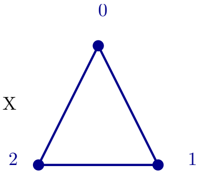

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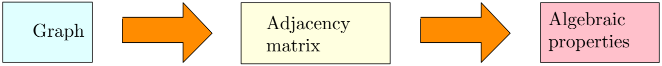

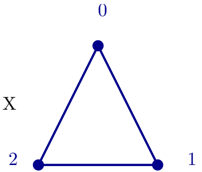

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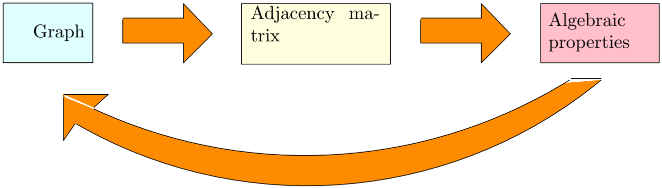

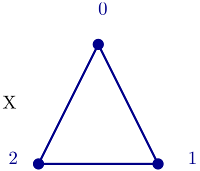

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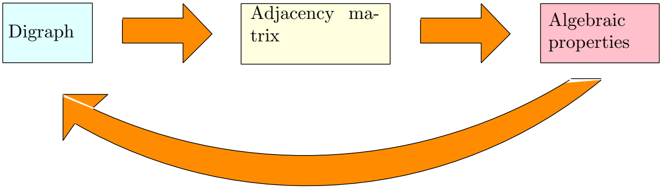

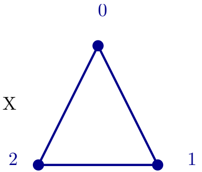

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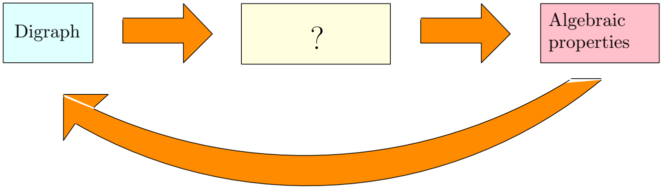
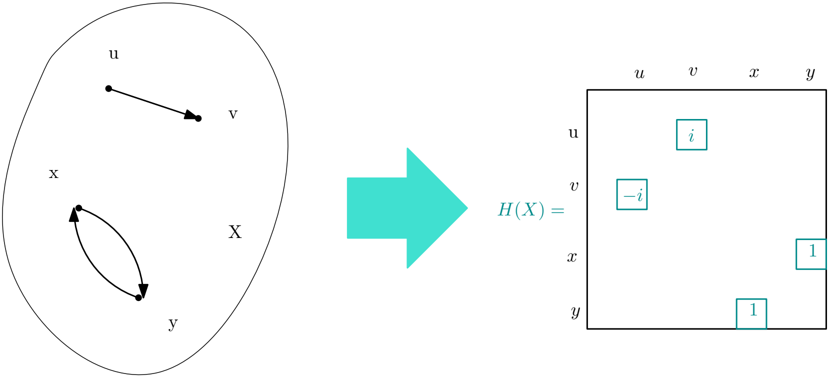


Hermitian adjacency matrix



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This matrix was independently defined by Liu and Li who used it to study energy of digraphs.

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- In particular, we may use eigenvalue interlacing.

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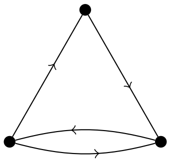
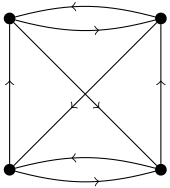
- The largest eigenvalue in absolute value could in fact be negative; we can have $|\lambda_n| > \lambda_1$.
- There does not appear to be a bound on the diameter of the digraph in terms of the number of distinct eigenvalues of the Hermitian adjacency matrix.
- In fact, there is an infinite family of weakly connected digraphs whose number of distinct eigenvalues is constant, but whose diameter goes to infinity.



K_3'



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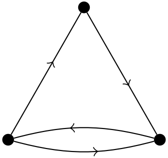

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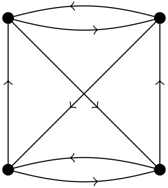

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These graphs have H-spectrum $\{-2, 1, 1\}$ and $\{-3, 1, 1\}$, resp.

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Theorem

For every digraph X we have

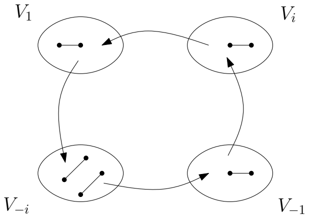
$$\lambda_1(X) \leq \rho(X) \leq 3\lambda_1(X).$$

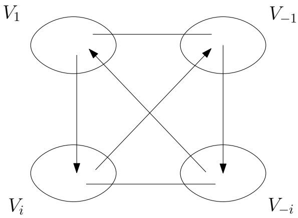
Both inequalities are tight.

We denote the underlying graph of X by $\Gamma(X)$.

Theorem

If X is a digraph, then $\rho(X) \leq \Delta(\Gamma(X))$. When X is weakly connected, the equality holds if and only if $\Gamma(X)$ is a $\Delta(\Gamma(X))$-regular graph and there exists a partition of $V(X)$ into four parts as in figures below.





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Note: in the latter case we can exactly which digraphs Y is isomorphic to.

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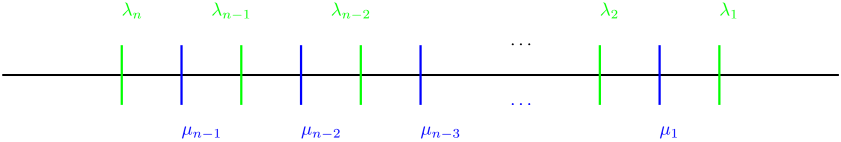

More generally, $\mu_1 \geq \cdots \geq \mu_m$ interlaces $\lambda_1 \geq \cdots \geq \lambda_n$ if

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Interlacing is a powerful tool that is used in the undirected case to find eigenvalue bounds on many graph properties.

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Theorem

If X has an independent set of size α, then $\eta^+(X) \geq \alpha$ and $\eta^-(X) \geq \alpha$.

Theorem (Gregory, Kirkland, Shader 93)

If X is an oriented graph of order n, then

$$\lambda_1(H(X)) \leq \cot\left(\frac{\pi}{2n}\right).$$

Equality holds if and only if X is switching-equivalent to T_n, the transitive tournament of order n.

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Corollary

If X is a digraph with an induced subdigraph that is switching equivalent to T_m, then $\lambda_1(H(X)) \geq \cot\left(\frac{\pi}{2m}\right)$.

Open problems

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- Other analogues of undirected spectral bounds.

Thanks!

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