WPI Physics Dept.
Intermediate Lab 2651
Physical Pendulum
(version 1.0)

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1 Introduction

The pendulum is one of the earliest and most important precision tools, which has found a variety of scientific applications. These include keeping time, measuring the gravitational acceleration, determining the density of the earth, and even determining the size of the earth. For example Eratosthenes used a hanging pendulum in the second century BC to determine the earth's radius. Arguably the pendulum is one of the corner stones of physics, because without a pendulum the development of the clock would have been unthinkable and without clocks it is impossible to do precision experimental physics. For example, Galileo had to rely on counting heartbeats for tracking time; however, heartbeats vary tremendously with exercise, and so in order to measure time Galileo had to be "calm and seated". Remarkably the pendulum was in use until the 1950s for cutting-edge research, and with the advent of MEMS again is in the forefront.

1.1 Mechanics

Since the pendulum executes circular motion, it is reasonable to perform an analysis based upon rotational mechanics: torque and mass moments. Consider the simple bar pendulum shown in Fig. 1a, whose axis of rotation is through the origin and along the $\hat{\mathbf{y}}$ direction. Define the first and second moments as

$$\chi_1 = \int \rho r dV \tag{1}$$

$$\chi_2 = \int \rho r^2 dV \ , \tag{2}$$

where the former is the product of the mass ($\int \rho dV$) with it's center ($r_{\text{com}} = \int \rho r dV / \int \rho dV$) and the latter is the moment of inertia. Note that here $r = \sqrt{x^2 + z^2}$ is the in-plane radial distance from the pivot. The torque due to the gravitational force acting on the pendulum is

$$\boldsymbol{\tau} = \int \mathbf{r} \times d\mathbf{F} = \int \rho \mathbf{r} \times \mathbf{g} dV \ . \tag{3}$$

As shown in the schematic Fig. 1a, the torque is

$$\boldsymbol{\tau} = -g\hat{\mathbf{y}} \int \rho \sin(\theta) r dV \approx -g \sin(\theta) \chi_1 \ , \ \text{where } \chi_1 = \hat{\mathbf{y}} \int \rho r dV \tag{4}$$

where the last approximation arises from the assumption that the angle does not vary significantly along the mass.

The torque leads to angular acceleration of the pendulum, which is

$$\tau = \chi_2 \ddot{\theta} \ . \tag{5}$$

Setting Eqns. (4) and (5) equal leads to the angular acceleration

$$\ddot{\theta} = -\frac{g\chi_1}{\chi_2} \sin(\theta) \ . \tag{6}$$

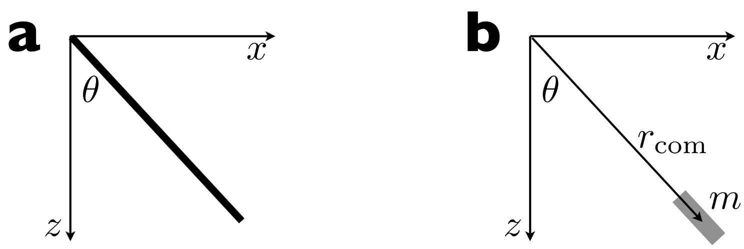

Figure 1: a) Schematic of a bar pendulum, at angle θ relative to the z-axis. b) An object whose center of mass is located at $\mathbf{r}_{\text{com}}$.

1.2 Small Amplitude Oscillations

For small angular displacements, the above can be approximated as

$$\ddot{\theta} \approx -\frac{g\chi_1}{\chi_2}\theta\ . \tag{7}$$

This is the famous harmonic differential equation, whose solution is sinusoidal:

$$\theta = \theta_0 \sin(\omega_0 t + \phi) \text{ where } \omega_0 = \sqrt{\frac{g\chi_1}{\chi_2}}\ , \tag{8}$$

and θ_0, ϕ are given by the initial conditions. The natural angular speed is defined as ω_0.

1.3 Large Amplitude Oscillations

For larger angles the non-linearities of Eq. (6) lead to a dependence of the period, T, on the instantaneous angle θ_0:

$$T = \frac{2\pi}{\omega_0}\left(1 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\theta_0}{2}\right) + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \sin^4\left(\frac{\theta_0}{2}\right) + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \sin^6\left(\frac{\theta_0}{2}\right) + \ldots\right)\ . \tag{9}$$

In particular the period increases with increasing amplitude, which is to be expected: Consider the case where the pendulum's release angle is close to $\theta_0 = \pi$ and the pendulum is almost "stuck" in the upside position. Fig. 2a shows the dependence of the period on the local apex angle, θ_0, as well as the analytical formula. The period increases severalfold, and for $\theta_0 = \pi$ it is infinite. Also note that for large amplitudes the relative velocity at the bottom of the swing is far greater than for small amplitudes. The analytical approximation holds for smaller angles, and for large angles additional terms of the series are necessary.

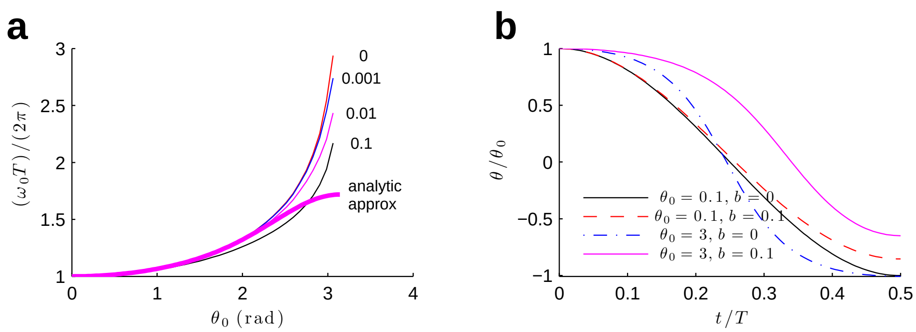

Figure 2: a) Normalized period of the pendulum as a function of the starting angle θ_0 for five different values of the friction coefficient. The analytical approximation is the first six terms from Eq. (9). b) The temporal dependence of the relative angular position for different starting angles and friction coefficients.

1.4 Frictional Losses

Finally, physical pendula are subject to frictional losses. These are proportional to the speed, thus the torque balance, Eq. (6), assumes an extra term

$$\ddot{\theta} + b\omega_0\dot{\theta} + \omega_0^2 \sin(\theta) = 0 \, , \tag{10}$$

where b is a dimensionless frictional factor. For the small-angle limit the solution is best found using complex exponentials,

$$\theta(t) = \theta_0 \exp(\lambda t) \tag{11}$$

where λ solves the quadratic equation

$$\lambda^2 + b\omega_0\lambda + \omega_0^2 = 0 \tag{12}$$

$$\lambda = \left(-b/2 \pm \sqrt{(b/2)^2 - 1}\right)\omega_0 \, . \tag{13}$$

The oscillations are

$$\theta(t) = \theta_0 \exp\left(-\frac{b\omega_0}{2}t\right)\cos(\sqrt{1-\frac{b^2}{4}}\omega_0 t + \phi) \, . \tag{14}$$

For small friction, $b^2 \ll 4$, (which is underdamping and expected for any reasonable pendulum), this approximates to

$$\lambda \approx \left(-\frac{b}{2} \pm i\right)\omega_0 \, , \tag{15}$$

and so the solution is

$$\theta(t) \approx \theta_0 \exp\left(-\frac{b\omega_0}{2}t\right)\cos(\omega_0 t + \phi) \, . \tag{16}$$

Fig. 2 has two simulations where the friction factor is $b = 0.1$. The figure shows that for greater amplitudes the pendulum's swing is diminished more than for smaller amplitudes.

The quality factor is defined as

$$Q = 2\pi \left(\frac{\text{Energy Stored}}{\text{Energy dissipated per cycle}} \right) = \frac{2\pi}{1 - \exp\left(-2\pi b\right)} \ , \tag{17}$$

which is a measure of how well the pendulum retains its energy. The last equality was obtained using Eq. (16). For underdamped systems, the quality factor is approximately the number of oscillations for the energy to fall off to $\exp(-2\pi) \approx 0.00187$ of its original energy. In the case of the pendulum, at the apex the energy is the potential energy, which is proportional to the height of the pendulum and is $\propto \sin\left(\theta_{\text{apex}}\right) \approx \theta_{\text{apex}}$.

1.5 Parallel Axis Theorem

It is useful to review the parallel axis theorem for the calculation of the second mass moment. For an object of mass, the center of mass is defined as $\mathbf{r}_{\text{com}} = \chi_1/m$. The second moment can be defined in terms of the center of mass,

$$\begin{aligned} \chi_2 = \int \mathbf{r}^2 dm = \int \left(\mathbf{r}_{\text{com}} + \left(\mathbf{r} - \mathbf{r}_{\text{com}}\right)\right)^2 dm = \\ r_{\text{com}}^2 m + 2\mathbf{r}_{\text{com}} \int \left(\mathbf{r} - \mathbf{r}_{\text{com}}\right) dm + \int \left(\mathbf{r} - \mathbf{r}_{\text{com}}\right)^2 dm = \\ r_{\text{com}}^2 m + I_{\text{com}} \ , \ \text{where } I_{\text{com}} = \int \left(\mathbf{r} - \mathbf{r}_{\text{com}}\right)^2 dm \ . \end{aligned} \tag{18}$$

In other words the moment of inertia about a certain point is the sum of the moments of inertia of the object about its center of mass and that of an equivalent point mass object - see Fig. 1b.

Calculations for the moment of inertia of standard objects about their center of mass are useful for determining the moment of inertia using the parallel axis theorem. Table 1 gives the moments for certain shapes

Table 1: The moments for certain objects about their centers of mass. R is the radius, and L is the (axial) length. The last expression is for the swinging rod.

sphere, radius R	$\frac{2}{5}mR^2$
slender rod, length L	$\frac{1}{12}mL^2$
axial cylinder, radius R	$\frac{1}{2}mR^2$
non-slender rod radius R, length L	$\frac{1}{4}mR^2 + \frac{1}{12}mL^2$

2 Pre-Lab Exercises

1. Calculate the relationship between ω, χ_1, χ_2 and g for a slender rod, length L, swinging about its end.
2. For what value of the friction factor does critical damping occur?
3. For what value of the friction factor does over-damping occur?
4. Explain the approximation in Eq. (4). (Hint: something got moved out of the integral...)
5. What is chaos and how does it apply to the double-pendulum?

3 Procedure

The new pendulum simply is a rod that is suspended from a knife-edge. The rod is threaded, so the pendulum's suspension point and hence $\chi_1,\ \chi_2$ can be modified. Simply by precisely measuring the lengths and the period of small-angle oscillation you can back out the earth's gravitational constant within a few percent or better.

The threaded rod pendulum is located in the hallway and is intended for small-angle oscillations and the determination of g. The ball-bearing pendulum next to it is intended for large angle oscillations. It has a lot more friction and is not adjustable.

For measuring the oscillations you will use a movie camera and bright LEDs attached to the pendulums. These will show up as bright spots in the movies, and image processing will be used to determine their position and hence their angle using the program `pendulum_video_small_angles.m`. This program requires some user inputs to identify the bright spots from the laser to determine the oscillation. At the end of this it will also try fit the damped oscillation Eq. (16). For the large amplitude oscillations (with the ball-bearings) you can confirm that the oscillation period depends on amplitude as predicted by Eq. (9). For this work with the program called `pendulum_large_oscillations.m`.

To precisely measure g you will use the dataset for different rod positions and the measured periods. You will want to vary the period from very short (knife edge almost at the middle) to very long (knife edge all the way at the top). You will fit these to the function

$$\omega_0 = c\sqrt{\frac{\chi_1/m}{\chi_2/m}} \tag{19}$$

Where $C = \sqrt{g}$, and

$$\begin{aligned}\frac{\chi_1}{m} &= L_0 + L_i + L_k \\ \frac{\chi_2}{m} &= \tfrac{I_{\text{com}}}{m} + (L_0 + L_i)^2 + \tfrac{I_k}{m}\ .\end{aligned}$$

Here the location of the center of mass is $L_0 + L_i$ where L_i is the length you measure and L_0 is some offset, and $L_k,\ I_k$ are (fudge) factors for the knife edge contributions.

If time permits investigate the double-pendulum's motion. Determine both normal modes.

4 Report Checklist

Your lab report should include at least the following parts:

1. Characterize the small amplitude oscillations of the rod pendulum for different supporting positions.
2. Measure the gravitational constant g from the pendulum data and compare with know values.
3. Characterize the oscillations of pendulum at large amplitudes.
4. Demonstrate the chaotic motion of the double pendulum.