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A Robust Cusum Test for SETAR-Type Nonlinearity in Time Series

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ABSTRACT
As a part of an effective self-exciting threshold autoregressive (SETAR) modeling methodology, it is important to identify processes exhibiting SETAR-type nonlinearity. A number of tests of nonlinearity have been developed in the literature. However, it has recently been shown that all these tests perform poorly for SETAR-type nonlinearity detection in the presence of additive outliers. In this paper, we develop an improved test for SETAR-type nonlinearity in time series. The test is an outlier-robust test based on the cumulative sums of ordered weighted residuals from generalized maximum likelihood fits. A Monte Carlo study confirms that the proposed test is competitive with existing tests for data from uncontaminated SETAR models and superior to them for SETAR data contaminated with additive outliers. Copyright © 2008 John Wiley & Sons, Ltd.

KEY WORDS GM estimation; outliers; additive outliers; innovational outliers

INTRODUCTION

A number of tests of time series nonlinearity have been developed in the literature, including those of Petruccelli and Davies (1986), Petruccelli (1990), Tsay (1986, 1989), Luukkonen *et al.* (1988), and Chan and Tong (1986, 1990). Chan and Ng (2004) have recently shown that all these tests perform poorly for self-exciting threshold autoregressive (SETAR)-type nonlinearity detection in the presence of additive outliers. Indeed, outliers may be responsible in at least some instances for apparent nonlinearity in time series data. Balke and Fomby (1994) give some examples in this sense using real economic data.

In this paper we propose a test for SETAR-type nonlinearity in time series which is robust against outliers. The proposed test, which we call CUSUM-GM, is a robust version of a modified Petruccelli–Davies (1986) test based on the cumulative sums of ordered weighted residuals from generalized maximum likelihood (GM) fits.

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Because the exact and asymptotic distributions of the CUSUM-GM test statistics are intractable, resampling is used to obtain the p-value for the test. Monte Carlo simulations show that, unlike the five tests considered by Chan and Ng, the CUSUM-GM test maintains its size in the presence of additive outliers.

BACKGROUND

This section details models and notation used in this paper.

The AR(p) model is defined as

$$x_t = \phi_0 + \sum_{l=1}^{p} \phi_l x_{t-l} + \epsilon_t, \quad t = p+1, \dots, T \tag{1}$$

where the ϵ_t are i.i.d. $N(0, \sigma_\epsilon^2)$ errors.

A straightforward extension of AR models is the class of AR-type nonlinear models defined as

$$x_t = f(x_{t-1}, x_{t-2}, \dots, x_{t-p}) + \epsilon_t \tag{2}$$

where $f: \mathbb{R}^p \to \mathbb{R}$ and ϵ_t are i.i.d. $N(0, \sigma_\epsilon^2)$ errors. We will refer to model (2) as GAR(p), for generalized AR(p).

A special case of the GAR(p) model is the SETAR(d; $p_1, p_2, \dots, p_k$) model, defined by

$$x_t = \phi_0^{(j)} + \sum_{l=1}^{p_j} \phi_l^{(j)} x_{t-l} + \epsilon_t \quad \text{if } x_{t-d} \in (r_{j-1}, r_j) \tag{3}$$

for $j = 1, 2, \dots, k$. The thresholds, r_i, satisfy $-\infty = r_0 < r_1 < \dots < r_k = \infty$, d, k and $p_1, p_2, \dots, p_k$ are positive integers, and ϵ_t are i.i.d. $N(0, \sigma_\epsilon^2)$ random variables with $\sigma_\epsilon^2 < \infty$. We say that the process is in the jth regime when $r_{j-1} < x_{t-\mathrm{d}} \leq r_j$.

We say that y_t is a GAR(p) time series with additive outliers (AO) if y_t satisfies

$$y_t = x_t + v_t, \quad 1 \leq t \leq T \tag{4}$$

where x_t is as defined in (2) and the v_t are independent random variables, independent of the sequence x_t and having the mixture density $(1 - \gamma)\delta_0(\cdot) + \gamma\pi$, where $0 \leq \gamma \leq 1$, $\delta_0(\cdot)$ is a degenerate density at 0, and π follows a $N(0, \omega^2\sigma_\epsilon^2)$. γ represents the proportion of contamination in the time series. In general, it is assumed that γ is small ($\gamma \leq 0.1$) because it appears that outliers in time series are present only a small fraction of the time (Denby and Martin, 1979).

We say that y_t is a GAR(p) time series with innovational outliers (IO) if y_t satisfies

$$y_t = x_t, \quad 1 \leq t \leq T \tag{5}$$

where x_t is as defined in (2) and the ϵ_t are i.i.d. with density $\gamma\pi_1 + (1 - \gamma)\pi_2$, where π_1 and π_2 are independent with $\pi_1 \sim N(0, \sigma_\epsilon^2)$, and $\pi_2 \sim N(0, \Delta^2\sigma_\epsilon^2)$.

GM ESTIMATION

Throughout this section, we will assume the data follow the AR(p) model given by (1). This can be written in the obvious vector notation as

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$$x_t = \boldsymbol{x}'_{t-1}\boldsymbol{\phi} + \epsilon_t, \quad t = p+1,\ldots,T \tag{6}$$

The GM estimator for an AR(p) model can be defined as the solution of the minimization problem

$$\min_{\phi} \sum_{t=p+1}^{T} W(x_{t-1})\rho\left(x_t - \boldsymbol{x}'_{t-1}\boldsymbol{\phi}\right) \tag{7}$$

where $\rho(\cdot)$ is a symmetric robustifying loss function, and $W(\cdot)$ is a non-negative symmetric robustifying weight function (Denby and Martin, 1979).

If ρ is differentiable, with derivative ψ, $\hat{\phi}_{\text{GM}}$ is the solution of the system of equations

$$\begin{aligned} \sum_{t=p+1}^{T} W(\boldsymbol{x}_{t-1})\psi\left(x_t - \boldsymbol{x}'_{t-1}\hat{\boldsymbol{\phi}}_{\text{GM}}\right) &= 0, \\ \sum_{t=p+1}^{T} x_{t-l}W(\boldsymbol{x}_{t-1})\psi\left(x_t - \boldsymbol{x}'_{t-1}\hat{\boldsymbol{\phi}}_{\text{GM}}\right) &= 0, \quad l = 1,\ldots,p \end{aligned} \tag{8}$$

Following Beaton and Tukey (1974), we express (8) in terms of the weight function, $w(u) = \psi(u)/u$, as

$$\sum_{t=p+1}^{T} w(e_t)W(\boldsymbol{x}_{t-1})x_{t-1}\left(x_t - \boldsymbol{x}'_{t-1}\hat{\boldsymbol{\phi}}_{\text{GM}}\right) = 0 \tag{9}$$

where $e_t = x_t - \boldsymbol{x}'_{t-1}\hat{\boldsymbol{\phi}}_{\text{GM}}$ for $t = p + 1, \ldots, T$ are the residuals.

Equation (9) can be solved by the iteratively reweighted least squares (IRLS) method as follows:

1. Get an initial estimate of $\boldsymbol{\phi}$, say $\hat{\boldsymbol{\phi}}_0$, usually by ordinary least squares.
2. Given $\hat{\boldsymbol{\phi}}_0$, compute the initial weights as

$$W(\boldsymbol{x}_{t-1})w(e_t^0) = \frac{W(\boldsymbol{x}_{t-1})\psi(e_t^0)}{e_t^0}$$

   where $e_t^0 = x_t - \boldsymbol{x}'_{t-1}\hat{\boldsymbol{\phi}}_0$, for $t = p + 1, \ldots, T$.
3. For $j = 0$ to convergence do the following:
   - obtain the weighted LS estimator $\hat{\boldsymbol{\phi}}_{j+1}$ by regressing x_t on $\boldsymbol{x}_{t-1}$ with weights $W(\boldsymbol{x}_{t-1})w(e_t^j)$;
   - compute the residuals $e_t^{j+1} = x_t - \boldsymbol{x}'_{t-1}\hat{\boldsymbol{\phi}}_{j+1}$, for $t = p + 1, \ldots, T$.

Convergence can be defined in a number of ways: relative change in the estimates; relative change in the scaled residuals; relative change in weights; preselected number of steps.

A number of influence functions have been proposed. Two of the most commonly used are those from the Huber family (H) and from the bisquare family (B). The Huber family is defined by

$$\psi_{H,c}(u) = uI(|u| \le c) + cI(|u| > c) \tag{10}$$

and the bisquare family by

$$\psi_{B,c}(u) = u\left(1 - u^2/c^2\right)^2 I(|u| \le c) \tag{11}$$

where $I(\cdot)$ is an indicator function, and $c > 0$.

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An effective strategy for obtaining a GM estimate is as follows: the initial estimate is computed by using least squares estimation; then the GM estimate based on the Huber influence function (the GM-H estimate) is computed by the IRLS method described above; the corresponding GM-H estimate is used as a starting point for computing the GM estimate based on the bisquare influence function (the GM-B estimate), again using the IRLS method. The use of the Huber influence function ensures that a unique root of equation (9) is obtained and the choice of the bisquare influence function leads to a more robust estimator in the case of AO model (Denby and Martin, 1979).

M estimation is GM estimation done with $W(\cdot) \equiv 1$. Denby and Martin (1979) show that the M estimator is robust to IO outliers, but not to AO outliers. In fact, they show that M estimators can have asymptotic bias nearly as large as least squares estimators in the AO case. They also show that GM estimation is successful in reducing asymptotic bias when the data are contaminated with additive outliers.

The proposed test, which will be fully described in the next section, is based on the residuals obtained from the GM fit. In order to obtain the GM estimates, we need to define the $W(\cdot)$ and $\psi(\cdot)$ functions. Following Denby and Martin (1979), we take

$$\psi(u) = c_r s_r \psi_0\left(\frac{u}{c_r s_r}\right) \tag{12}$$

and generalizing their choice for a zero-mean AR(1), we define the weight function $W(\cdot)$ by

$$W(\boldsymbol{x}_{t-1}) = \prod_{i=1}^{p} \frac{g_0\left(\dfrac{x_{t-i} - M_x}{c_x s_x}\right)}{\dfrac{x_{t-i} - M_x}{c_x s_x}} \tag{13}$$

where M_x is a robust estimate of location for x_t, s_x and s_r are robust estimates of scale for x_t and ϵ_t respectively, and g_0, ψ_0 are influence functions. In our application $M_x = \text{median}(x_i)$ and

$$s_x = \text{median}(|x_i - M_x|)/0.6745 \tag{14}$$

are computed only once, while s_r is obtained from the residuals (e_i) at each step of the IRLS procedure, using the formula

$$s_r = \text{median}(|e_i - \text{median}(e_i)|)/0.6745 \tag{15}$$

We also choose $g_0 = \psi_0$ to be either the Huber or bisquare influence function, and denote the resulting estimators GM-H and GM-B. For the GM-H estimator, we will take $c_x = c_{H,x} = 1$ for g_0 and $c_r = c_{H,r} = 1.5$ for ψ_0. For the GM-B estimator, we will take c is defined as $c_x = c_{B,x} = 3.9$ for g_0 and $c_r = c_{B,r} = 1.5$ for ψ_0. These parameters are chosen to get 95% asymptotic efficiency on the standard normal and non-normal distributions simultaneously.

The GM-B estimator is preferred to the GM-H estimator because of its superiority in the AO model along with its reasonable robustness in the IO model (Denby and Martin, 1979). This leads us to choose GM-B estimation for the proposed test. Using equation (8), we compute the GM-B estimator as follows: we get an initial estimate by using ordinary least squares; then we compute the GM estimator based on the Huber influence function (the GM-H estimate) by choosing $g_0(\cdot) = \psi_o(\cdot)$

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$= \psi_H(\cdot)$; then the GM-H estimate is used as a starting point for GM-B estimation for which $g_0(\cdot) = \psi_o(\cdot) = \psi_B(\cdot)$. For our computations, we consider that the convergence is obtained when the relative change in the parameter estimates is less than 10^{-4}.

CUSUM-GM TEST FOR SETAR-TYPE NONLINEARITY

In order to obtain a robust test for SETAR-type nonlinearity, we apply GM estimation to the ordered autoregression defining the CUSUM test of Petruccelli and Davies (1986).

The proposed CUSUM-GM test statistic is given by

$$Z = \sup_{0 \le \lambda \le 1} \left| \frac{1}{A\sqrt{T-p}} Z_{\mathrm{CGM}}^{(T)}(\lambda) \right| \tag{16}$$

where T is the number of observation in the time series, A is defined as

$$\sqrt{\frac{\sum_{t=p+1}^{T} W^2(\boldsymbol{x}_{t-1})\psi^2\left(x_t - \boldsymbol{x}'_{t-1}\hat{\phi}_{\mathrm{GM}}\right)}{T-p}} \tag{17}$$

and

$$Z_{\mathrm{CGM}}^{(T)}(\lambda) = \sum_{t=p+1}^{[T\lambda]} W(\boldsymbol{x}_{t-1})\psi\left(x_t - \boldsymbol{x}'_{t-1}\hat{\phi}_{\mathrm{GM}}\right) \tag{18}$$

where $\hat{\boldsymbol{\phi}}_{\mathrm{GM}}$ is the GM estimate defined in (7).

Note that we use the ordinary residuals rather than the predictive residuals used in Petruccelli and Davies (1986). This is because in the absence of recursive updating operations the amount of computation involved in trying to assess the performance of the test through Monte Carlo simulations is prohibitive.

Because the exact and asymptotic distributions of Z under the assumption of linearity are intractable, we use resampling to obtain a p-value.

The algorithm is as follows:

1. Compute $\hat{\boldsymbol{\phi}}_{\mathrm{GM}}$ and $\hat{\sigma}_\epsilon$, where $\hat{\sigma}_\epsilon$ is calculated using (15).
2. Compute the CUSUM-GM test statistic Z^*.
3. For $j = 1$ to 1000:
   (i) generate an AR(p) ~ $\hat{\boldsymbol{\phi}}_{\mathrm{GM}}$, $\hat{\sigma}_\epsilon$, T;
   (ii) compute the CUSUM-GM statistic Z_j^*.
4. Compute the empirical p-value as the proportion of $Z_j^* \ge Z^*$.

An R program for computing the test is available from the authors upon request.

SIMULATION RESULTS

A Monte Carlo simulation study was conducted to evaluate the performance of the CUSUM-GM test on simulated data and compare that performance with corresponding published results for five

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other nonlinearity tests: C-PD, the CUSUM test of Petruccelli and Davies (1986); RC, the reverse CUSUM test of Petruccelli (1990); F, the F test of Tsay (1989); LM, the Lagrange multiplier test of Luukkonen *et al.* (1988); and LR, the likelihood ratio test of Chan and Tong (1986). Chan and Ng (2004) provide a good description of these tests.

The simulation results for all but the CUSUM-GM test were obtained from Chan and Ng (2004), and are reproduced here for comparison. The model parameters for the CUSUM-GM test were chosen to match those simulations done by Chan and Ng. Namely, all were from a SETAR(1;1,1) with $r_1 = 0$ and $\phi_1^{(1)} = 0.5$, and the effect of varying $\phi_1^{(2)} = 0.5, 0.8, -0.3$ and $\phi_0^{(1)} = \phi_0^{(2)} = 0, 1$ were considered.

All of the simulations were based on 1000 replications of the AO or IO model with $\gamma = 0.05$, and $\sigma_\epsilon^2 = 1$. Sample sizes $T = 100$ and 200 were used. The first 1500 observations in each replication were discarded to avoid dependence on the initial value, which was set to zero. The outlier magnitudes considered were $\omega = 0, 3, 6, 10$ for the AO model and $\Delta = 1, 3, 6, 10$ for the IO model. It should be noted that $\omega = 0$ under AO and $\Delta = 1$ under IO correspond to the no-outlier case. We assumed that $p = 1$ and $d = 1$ are known. We compared the tests both in terms of size and power when the level of significance is fixed at a nominal $\alpha = 0.05$. The results, displayed in Tables I–IV, show the following.

Table I. The empirical frequencies of rejecting the null hypothesis of linearity at the nominal 5% level, additive outlier case; sample size $T = 100$

Parameters					Tests for SETAR-type nonlinearity					
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	C-GM	C-PD	RC	LR	LM	F
0.0	0.5	0.0	0.5	0	0.038	0.041	0.013	0.056	0.033	0.043
				3	0.069	0.160	0.034	0.129	0.212	0.162
				6	0.081	0.289	0.083	0.231	0.333	0.275
				10	0.059	0.372	0.126	0.222	0.268	0.246
1.0	0.5	1.0	0.5	0	0.036	0.032	0.013	0.047	0.039	0.036
				3	0.061	0.163	0.044	0.154	0.247	0.182
				6	0.066	0.295	0.086	0.260	0.349	0.293
				10	0.064	0.365	0.116	0.215	0.263	0.242
0.0	0.5	0.0	0.8	0	0.111	0.093	0.039	0.119	0.106	0.095
				3	0.093	0.328	0.130	0.318	0.460	0.298
				6	0.107	0.545	0.267	0.561	0.693	0.530
				10	0.119	0.580	0.319	0.512	0.582	0.505
1.0	0.5	1.0	0.8	0	0.057	0.043	0.020	0.059	0.048	0.039
				3	0.058	0.276	0.099	0.286	0.477	0.307
				6	0.092	0.569	0.323	0.646	0.791	0.688
				10	0.048	0.659	0.402	0.665	0.754	0.723
0.0	0.5	0.0	–0.3	0	0.270	0.157	0.280	0.466	0.471	0.430
				3	0.252	0.127	0.172	0.284	0.253	0.309
				6	0.261	0.162	0.089	0.160	0.144	0.204
				10	0.282	0.245	0.075	0.105	0.101	0.162
1.0	0.5	1.0	–0.3	0	0.175	0.338	0.205	0.341	0.381	0.393
				3	0.180	0.166	0.102	0.210	0.188	0.202
				6	0.158	0.123	0.116	0.141	0.118	0.153
				10	0.181	0.142	0.102	0.071	0.077	0.102

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Table II. The empirical frequencies of rejecting the null hypothesis of linearity at the nominal 5% level, additive outlier case; sample size $T = 200$

Parameters					Tests for SETAR-type nonlinearity					
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	C-GM	C-PD	RC	LR	LM	F
0.0	0.5	0.0	0.5	0	0.043	0.028	0.013	0.044	0.026	0.033
				3	0.063	0.334	0.100	0.259	0.477	0.346
				6	0.076	0.634	0.233	0.532	0.658	0.647
				10	0.055	0.576	0.243	0.415	0.394	0.476
1.0	0.5	1.0	0.5	0	0.044	0.025	0.010	0.039	0.031	0.029
				3	0.044	0.330	0.083	0.253	0.496	0.363
				6	0.063	0.610	0.227	0.539	0.652	0.619
				10	0.065	0.597	0.251	0.420	0.405	0.498
0.0	0.5	0.0	0.8	0	0.188	0.143	0.121	0.209	0.207	0.204
				3	0.173	0.601	0.302	0.507	0.745	0.546
				6	0.178	0.857	0.590	0.864	0.932	0.874
				10	0.200	0.848	0.643	0.838	0.796	0.844
1.0	0.5	1.0	0.8	0	0.036	0.042	0.018	0.057	0.048	0.040
				3	0.060	0.455	0.184	0.435	0.703	0.515
				6	0.085	0.825	0.590	0.884	0.971	0.930
				10	0.065	0.911	0.741	0.937	0.937	0.964
0.0	0.5	0.0	–0.3	0	0.530	0.299	0.495	0.771	0.796	0.765
				3	0.496	0.218	0.321	0.517	0.427	0.567
				6	0.482	0.352	0.146	0.315	0.181	0.429
				10	0.482	0.358	0.082	0.153	0.110	0.249
1.0	0.5	1.0	–0.3	0	0.396	0.655	0.490	0.636	0.722	0.731
				3	0.325	0.217	0.232	0.330	0.271	0.376
				6	0.309	0.100	0.113	0.141	0.087	0.171
				10	0.324	0.100	0.078	0.076	0.069	0.112

Outlier-free case

There is no test unacceptable for size in this case. The power of C-GM is comparable with that of the best of the other tests for $\phi_1^{(1)} = 0.5$, $\phi_1^{(2)} = 0.8$. For $\phi_0^{(1)} = 0.0$, $\phi_1^{(1)} = 0.5$, $\phi_0^{(2)} = 0.0$, $\phi_1^{(2)} = -0.3$, it performs comparably with RC and outperforms only C-PD. For $\phi_0^{(1)} = 1.0$, $\phi_1^{(1)} = 0.5$, $\phi_0^{(2)} = 1.0$, $\phi_1^{(2)} = -0.3$, it either performs worst ($T = 200$), or tied for worst ($T = 100$).

Additive outlier case

C-GM maintains an acceptable size for all ω. Tests C-PD, LR, LM, and F perform unacceptably due to loss of size for $\omega > 0$. RC maintains size better, but performs unacceptably for $T = 100$, $\omega = 10$, and for $T = 200$, $\omega \geq 6$. For the four nonlinear parameter sets, the power of C-GM is largely unaffected by ω values, in marked contrast to the other tests. For $\phi_1^{(1)} = 0.5$, $\phi_1^{(2)} = 0.8$, the power of C-GM is much lower than that of the other tests, whereas for $\phi_1^{(1)} = 0.5$, $\phi_1^{(1)} = -0.3$, the power of C-GM is comparable to or better than that of the other tests. Further, it should be noted that the low power of the test for $\phi_1^{(1)} = 0.5$, $\phi_1^{(2)} = 0.8$ is expected because the series is essentially linear for this configuration since the proportion of time spent in the first regime is negligible.

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Table III. The empirical frequencies of rejecting the null hypothesis of linearity at the nominal 5% level, innovational outlier case; sample size $T = 100$

Parameters					Tests for SETAR-type nonlinearity					
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	Δ	C-GM	C-PD	RC	LR	LM	F
0.0	0.5	0.0	0.5	1	0.057	0.036	0.014	0.046	0.032	0.037
				3	0.069	0.034	0.015	0.035	0.030	0.035
				6	0.062	0.059	0.032	0.038	0.035	0.052
				10	0.069	0.153	0.040	0.037	0.041	0.073
1.0	0.5	1.0	0.5	1	0.054	0.034	0.009	0.045	0.035	0.034
				3	0.046	0.036	0.009	0.043	0.034	0.033
				6	0.072	0.057	0.024	0.039	0.038	0.049
				10	0.062	0.128	0.053	0.039	0.040	0.069
0.0	0.5	0.0	0.8	1	0.110	0.087	0.033	0.099	0.102	0.088
				3	0.116	0.089	0.049	0.107	0.098	0.113
				6	0.146	0.096	0.055	0.095	0.105	0.133
				10	0.147	0.162	0.077	0.099	0.087	0.146
1.0	0.5	1.0	0.8	1	0.061	0.023	0.008	0.044	0.043	0.026
				3	0.068	0.023	0.014	0.039	0.038	0.035
				6	0.067	0.052	0.032	0.055	0.056	0.067
				10	0.074	0.165	0.049	0.074	0.081	0.110
0.0	0.5	0.0	–0.3	1	0.260	0.149	0.243	0.444	0.471	0.429
				3	0.269	0.141	0.242	0.444	0.505	0.490
				6	0.246	0.126	0.256	0.549	0.598	0.575
				10	0.259	0.212	0.244	0.558	0.654	0.626
1.0	0.5	1.0	–0.3	1	0.202	0.366	0.194	0.367	0.411	0.407
				3	0.220	0.422	0.287	0.444	0.493	0.512
				6	0.249	0.427	0.350	0.533	0.593	0.620
				10	0.231	0.435	0.362	0.565	0.646	0.669

Innovational outlier case

C-GM maintains an acceptable size for all Δ, as do all other tests, except for C-PD when $\Delta = 10$. The power of C-GM is comparable to that of the other tests for $\phi_1^{(1)} = 0.5$, $\phi_1^{(2)} = 0.8$. For $\phi_0^{(1)} = 0.0$, $\phi_1^{(1)} = 0.5$, $\phi_0^{(2)} = 0.0$, $\phi_1^{(2)} = -0.3$, C-GM outperforms C-PD, and is outperformed by LR, LM, and F. For $T = 100$, it outperforms RC. For $T = 200$ and $\Delta = 3$, if performs comparably with RC, while for $\Delta = 6$ and 10 it outperforms RC. For $\phi_0^{(1)} = 1.0$, $\phi_1^{(1)} = 0.5$, $\phi_0^{(2)} = 1.0$, $\phi_1^{(2)} = -0.3$, all other tests outperform C-GM.

If in model (3) we know that $\phi_0^{(j)} = 0, j = 1, \ldots, k$, then we may improve the power of the C-GM test by fitting AR models without intercepts. Tables V–VIII show the results of a simulation study in which the performance of the C-GM statistic is compared with that of C-GM*, the C-GM statistic obtained by fitting AR models without intercepts. In this setting, C-GM* outperforms the other tests considered.

CONCLUSION

We have developed a test for nonlinearity in time series based on Cusums of weighted residuals from GM autoregressive fits. Simulation results show that the test is effective in maintaining the nominal significance levels while exhibiting reasonable power in the presence of additive and innovational outliers.

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Table IV. The empirical frequencies of rejecting the null hypothesis of linearity at the nominal 5% level, innovational outlier case; sample size $T = 200$

Parameters					Tests for SETAR-type nonlinearity					
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	Δ	C-GM	C-PD	RC	LR	LM	F
0.0	0.5	0.0	0.5	1	0.048	0.039	0.016	0.059	0.038	0.053
				3	0.061	0.042	0.015	0.042	0.040	0.038
				6	0.053	0.055	0.032	0.038	0.038	0.047
				10	0.065	0.064	0.030	0.039	0.050	0.076
1.0	0.5	1.0	0.5	1	0.051	0.044	0.010	0.059	0.048	0.052
				3	0.043	0.046	0.011	0.037	0.042	0.041
				6	0.063	0.040	0.023	0.048	0.041	0.054
				10	0.060	0.057	0.024	0.023	0.044	0.054
0.0	0.5	0.0	0.8	1	0.197	0.197	0.099	0.212	0.252	0.252
				3	0.210	0.190	0.121	0.224	0.229	0.260
				6	0.230	0.158	0.126	0.196	0.221	0.288
				10	0.249	0.169	0.162	0.227	0.274	0.351
1.0	0.5	1.0	0.8	1	0.048	0.034	0.015	0.053	0.048	0.031
				3	0.058	0.058	0.010	0.051	0.049	0.054
				6	0.055	0.118	0.031	0.068	0.090	0.131
				10	0.074	0.231	0.082	0.118	0.146	0.210
0.0	0.5	0.0	–0.3	1	0.533	0.320	0.507	0.770	0.792	0.766
				3	0.514	0.309	0.524	0.840	0.851	0.841
				6	0.493	0.263	0.451	0.882	0.933	0.914
				10	0.506	0.231	0.400	0.905	0.933	0.923
1.0	0.5	1.0	–0.3	1	0.410	0.627	0.502	0.627	0.700	0.710
				3	0.443	0.706	0.634	0.779	0.809	0.850
				6	0.437	0.756	0.726	0.893	0.931	0.939
				10	0.479	0.628	0.635	0.921	0.952	0.954

Table V. The empirical frequencies of rejecting the null hypothesis of linearity at the nominal 5% level, additive outlier case; sample size $T = 100$

Parameters					Tests for SETAR-type nonlinearity						
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	C-GM*	C-GM	C-PD	RC	LR	LM	F
0.0	0.5	0.0	0.5	0	0.052	0.038	0.041	0.013	0.056	0.033	0.043
				3	0.056	0.069	0.160	0.034	0.129	0.212	0.162
				6	0.056	0.081	0.289	0.083	0.231	0.333	0.275
				10	0.052	0.059	0.372	0.126	0.222	0.268	0.246
0.0	0.5	0.0	0.8	0	0.194	0.111	0.093	0.039	0.119	0.106	0.095
				3	0.223	0.093	0.328	0.130	0.318	0.460	0.298
				6	0.210	0.107	0.545	0.267	0.561	0.693	0.530
				10	0.206	0.119	0.580	0.319	0.512	0.582	0.505
0.0	0.5	0.0	–0.3	0	0.644	0.270	0.157	0.280	0.466	0.471	0.430
				3	0.580	0.252	0.127	0.172	0.284	0.253	0.309
				6	0.534	0.261	0.162	0.089	0.160	0.144	0.204
				10	0.551	0.282	0.245	0.075	0.105	0.101	0.162

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Table VI. The empirical frequencies of rejecting the null hypothesis of linearity at the nominal 5% level, additive outlier case; sample size $T = 200$

Parameters					Tests for SETAR-type nonlinearity						
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	ω	C-GM*	C-GM	C-PD	RC	LR	LM	F
0.0	0.5	0.0	0.5	0	0.050	0.043	0.028	0.013	0.044	0.026	0.033
				3	0.050	0.063	0.334	0.100	0.259	0.477	0.346
				6	0.058	0.076	0.634	0.233	0.532	0.658	0.647
				10	0.051	0.055	0.576	0.243	0.415	0.394	0.476
0.0	0.5	0.0	0.8	0	0.377	0.188	0.143	0.121	0.209	0.207	0.204
				3	0.416	0.173	0.601	0.302	0.507	0.745	0.546
				6	0.408	0.178	0.857	0.590	0.864	0.932	0.874
				10	0.376	0.200	0.848	0.643	0.838	0.796	0.844
0.0	0.5	0.0	–0.3	0	0.968	0.530	0.299	0.495	0.771	0.796	0.765
				3	0.926	0.496	0.218	0.321	0.517	0.427	0.567
				6	0.933	0.482	0.352	0.146	0.315	0.181	0.429
				10	0.915	0.482	0.358	0.082	0.153	0.110	0.249

Table VII. The empirical frequencies of rejecting the null hypothesis of linearity at the nominal 5% level, innovational outlier case; sample size $T = 100$

Parameters					Tests for SETAR-type nonlinearity						
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	Δ	C-GM*	C-GM	C-PD	RC	LR	LM	F
0.0	0.5	0.0	0.5	1	0.056	0.057	0.036	0.014	0.046	0.032	0.037
				3	0.064	0.069	0.034	0.015	0.035	0.030	0.035
				6	0.055	0.062	0.059	0.032	0.038	0.035	0.052
				10	0.046	0.069	0.153	0.040	0.037	0.041	0.073
0.0	0.5	0.0	0.8	1	0.178	0.110	0.087	0.033	0.099	0.102	0.088
				3	0.175	0.116	0.089	0.049	0.107	0.098	0.113
				6	0.217	0.146	0.096	0.055	0.095	0.105	0.133
				10	0.232	0.147	0.162	0.077	0.099	0.087	0.146
0.0	0.5	0.0	–0.3	1	0.611	0.260	0.149	0.243	0.444	0.471	0.429
				3	0.550	0.269	0.141	0.242	0.444	0.505	0.490
				6	0.487	0.246	0.126	0.256	0.549	0.598	0.575
				10	0.444	0.259	0.212	0.244	0.558	0.654	0.626

Table VIII. The empirical frequencies of rejecting the null hypothesis of linearity at the nominal 5% level, innovational outlier case; sample size $T = 200$

Parameters					Tests for SETAR-type nonlinearity						
$\phi_0^{(1)}$	$\phi_1^{(1)}$	$\phi_0^{(2)}$	$\phi_1^{(2)}$	Δ	C-GM*	C-GM	C-PD	RC	LR	LM	F
0.0	0.5	0.0	0.5	1	0.054	0.048	0.039	0.016	0.059	0.038	0.053
				3	0.052	0.061	0.042	0.015	0.042	0.040	0.038
				6	0.054	0.053	0.055	0.032	0.038	0.038	0.047
				10	0.063	0.065	0.064	0.030	0.039	0.050	0.076
0.0	0.5	0.0	0.8	1	0.380	0.197	0.197	0.099	0.212	0.252	0.252
				3	0.411	0.210	0.190	0.121	0.224	0.229	0.260
				6	0.427	0.230	0.158	0.126	0.196	0.221	0.288
				10	0.478	0.249	0.169	0.162	0.227	0.274	0.351
0.0	0.5	0.0	–0.3	1	0.970	0.533	0.320	0.507	0.770	0.792	0.766
				3	0.935	0.514	0.309	0.524	0.840	0.851	0.841
				6	0.897	0.493	0.263	0.451	0.882	0.933	0.914
				10	0.902	0.506	0.231	0.400	0.905	0.933	0.923

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