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ON THE NULL RECURRENCE AND TRANSIENCE OF A FIRST-ORDER SETAR MODEL

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Abstract

We consider the SETAR(l; $1, \cdots, 1$) model:

$$X_t = \phi(0,j) + \phi(1,j)X_{t-1} + \varepsilon_t(j), \qquad r_{j-1} < X_{t-1} \leqq r_j$$

where $-\infty = r_0 < r_1 < \cdots < r_l = \infty$ and for each j $\{\varepsilon_t(j)\}$ forms an i.i.d. zero-mean error sequence independent of $\{\varepsilon_t(i)\}$ for $i \neq j$ and having a density positive on the real line. Chan et al. (1985) obtained the region of the parameter space on which the process is ergodic, and showed the process to be transient on a subset of the remainder. They conjectured that the process was null recurrent everywhere else. In this paper we show that conjecture to be incorrect and under the assumption of finite variance of the error distributions we resolve the remaining questions of transience or null recurrence for this process.

NON-LINEAR TIME SERIES; ERGODICITY; MARKOV CHAINS

1. Introduction

In this paper we consider the first-order SETAR(l; $1, \cdots, 1$) model

$$X_t = \phi(0,j) + \phi(1,j)X_{t-1} + \varepsilon_t(j), \qquad r_{j-1} < X_{t-1} \leqq r_j \tag{1.1}$$

where $-\infty = r_0 < r_1 < \cdots < r_l = \infty$. We assume the following two conditions on the error distributions.

C1. For each $j = 1, \cdots, l$ $\{\varepsilon_t(j)\}$ forms an i.i.d. zero-mean error sequence independent of $\{\varepsilon_t(i)\}$ for $i \neq j$.

C2. For each $j = 1, \cdots, l$, $\varepsilon_t(j)$ has a density positive on the real line.

For the no-intercept case ($\phi(0,j) = 0$, $j = 1, \cdots, l$), Petruccelli and Woolford (1984) showed a necessary and sufficient condition for ergodicity of the process to be:

$$\phi(1,1) < 1, \quad \phi(1,l) < 1, \quad \phi(1,1)\phi(1,l) < 1.$$

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They showed the process to be transient on the complement of the closure of this set and conjectured it to be null recurrent on the boundary.

Chan et al. (1985) considered the general model (1.1). They obtained as necessary and sufficient conditions for egodicity (2.1)–(2.5) below. They also showed that the process is transient under conditions (2.11), (2.12) and (2.15), and conjectured that the process is null recurrent otherwise.

In this paper we show the conjecture of Chan et al. to be incorrect, and under the additional assumption of finite error distribution variance in regions 1 and l we resolve the remaining questions of transience or null recurrence for this process.

For physical motivations for and applications of SETAR models, and for a discussion of types of non-linear phenomena which such models can capture, we refer the reader to Tong (1983).

2. Results

Throughout the rest of this paper, except where otherwise stated, we assume that in addition to the conditions on the error distributions detailed in Section 1, the following holds.

C3. $E(\varepsilon_t^2(j)) < \infty, j = 1$ *and* $j = l$.

The main result of the paper is the following theorem.

Theorem. *For model* (1.1) *with conditions* $C1$–$C3$ *holding, the following characterize the ergodicity, null recurrence or transience of the process*:

I. *The process is ergodic if and only if one of* (2.1)–(2.5) *holds*:

$$\phi(1,1) < 1, \quad \phi(1,l) < 1, \quad \phi(1,1)\phi(1,l) < 1, \tag{2.1}$$

$$\phi(1,1) = 1, \quad \phi(1,l) < 1, \quad \phi(0,1) > 0, \tag{2.2}$$

$$\phi(1,1) < 1, \quad \phi(1,l) = 1, \quad \phi(0,l) < 0, \tag{2.3}$$

$$\phi(1,1) = \phi(1,l) = 1, \quad \phi(0,l) < 0 < \phi(0,1), \tag{2.4}$$

$$\phi(1,1) < 0, \quad \phi(1,1)\phi(1,l) = 1, \quad \phi(0,l) + \phi(1,l)\phi(0,1) > 0. \tag{2.5}$$

II. *The process is null recurrent if and only if one of* (2.6)–(2.10) *holds*:

$$\phi(1,1) < 1, \quad \phi(1,l) = 1, \quad \phi(0,l) = 0, \tag{2.6}$$

$$\phi(1,1) = 1, \quad \phi(1,l) < 1, \quad \phi(0,1) = 0, \tag{2.7}$$

$$\phi(1,1) = \phi(1,l) = 1, \quad \phi(0,l) = 0, \quad \phi(0,1) \geqq 0, \tag{2.8}$$

$$\phi(1,1) = \phi(1,l) = 1, \quad \phi(0,l) < 0, \quad \phi(0,1) = 0, \tag{2.9}$$

$$\phi(1,1) < 0, \quad \phi(1,1)\phi(1,l) = 1, \quad \phi(0,l) + \phi(1,l)\phi(0,1) = 0. \tag{2.10}$$

III. *The process is transient if and only if one of* (2.11)–(2.16) *holds*:

$$\phi(1,1) > 1, \tag{2.11}$$

(2.12) $$\phi(1, l) > 1,$$

(2.13) $$\phi(1, 1) = 1, \quad \phi(1, l) \leqq 1, \quad \phi(0, 1) < 0,$$

(2.14) $$\phi(1, 1) \leqq 1, \quad \phi(1, l) = 1, \quad \phi(0, l) > 0,$$

(2.15) $$\phi(1, 1) < 0, \quad \phi(1, 1)\phi(1, l) > 1,$$

(2.16) $$\phi(1, 1) < 0, \quad \phi(1, 1)\phi(1, l) = 1, \quad \phi(0, l) + \phi(1, l)\phi(0, 1) < 0.$$

The proof of the theorem will proceed by a series of lemmas in the next section. We remark here that:

1. (2.1)–(2.5) and (2.11), (2.12) and (2.15) were derived in Chan et al. (1985) and do not depend on condition C3.
2. (2.13) and (2.14) do not depend on condition C3.
3. The theorem shows the conjecture of Petruccelli and Woolford (1984) to be correct under condition C3.
4. Condition C2 ensures that the process is μ-irreducible for μ taken to be Lebesgue measure on the real line. This is needed to invoke Tweedie's (1976) criteria for null recurrence and transience.

3. Proofs

Lemma 3 below proves that (2.6)–(2.10) define the region of null recurrence for (1.1). Lemma 4 below proves the transience of (1.1) in region (2.16). We begin with two technical lemmas used in the proof of Lemma 3.

Lemma 1. *Let η be a random variable, s a positive number and t any real number. Then for any $A \subseteq \{\eta : s + t\eta > 0\}$, and $B \subseteq \{\eta : -s + t\eta > 0\}$,*
(i) $E[\ln(s + t\eta)I_A] \leqq P(A)\ln(s) + (t/s)E[\eta I_A] - (t^2/(2s^2))E[\eta^2 I_{[A \cap \{\eta : t\eta < 0\}]}]$,
(ii) $E[\ln(-s + t\eta)I_B] \leqq P(B)(\ln(s) - 2) + (t/s)E[\eta I_B]$.

Proof. (i) For all $x > -1$, $\ln(1 + x) \leqq x - (x^2/2)I_{[x<0]}$. Thus

$$\begin{aligned}\ln(s + t\eta)I_A &= [\ln(s) + \ln(1 + t\eta/s)]I_A \\ &\leqq [\ln(s) + t\eta/s - ((t\eta)^2/(2s^2))I_{[\eta : t\eta < 0]}]I_A,\end{aligned}$$

and taking expectations gives the result.
(ii) For all $x > 1$, $\ln(-1 + x) \leqq x - 2$. Thus

$$\begin{aligned}\ln(-s + t\eta)I_B &= [\ln(s) + \ln(-1 + t\eta/s)]I_B \\ &\leqq (\ln(s) + t\eta/s - 2)I_B,\end{aligned}$$

and taking expectations gives the result.

Lemma 2. *Let η be a random variable with distribution function G and finite variance, let t, c, u_2, and v_2 be positive numbers, and let $s_1 \geqq s_2$ and u_1, v_1, s be real numbers. Then*

(i)
$$\lim_{x \to -\infty} xE[\eta I_{[\eta < s+tx]}] = \lim_{x \to \infty} xE[\eta I_{[\eta > s+tx]}] = 0.$$

Furthermore, if $E(\eta) = 0$, *then*

$$\lim_{x \to -\infty} xE[\eta I_{[\eta > s+tx]}] = \lim_{x \to \infty} xE[\eta I_{[\eta < s+tx]}] = 0.$$

(ii)
$$\lim_{x \to -\infty} x^2[-G(s_1 + tx)\ln(u_1 - u_2 x) + G(s_2 + tx)(\ln(v_1 - v_2 x) - c)] \leqq 0.$$

(iii)
$$\lim_{x \to \infty} x^2[-(1 - G(s_2 + tx))\ln(v_1 + v_2 x) + (1 - G(s_1 + tx))(\ln(u_1 + u_2 x) - c)] \leqq 0.$$

Proof. (i) The first line follows from

$$0 \leqq \lim_{x \to \infty} (s + tx) \int_{s+tx}^{\infty} \eta dG(\eta) \leqq \lim_{x \to \infty} \int_{s+tx}^{\infty} \eta^2 dG(\eta) = 0,$$

and

$$0 \leqq \lim_{x \to -\infty} (s + tx) \int_{-\infty}^{s+tx} \eta dG(\eta) \leqq \lim_{x \to -\infty} \int_{-\infty}^{s+tx} \eta^2 dG(\eta) = 0.$$

If $E(\eta) = 0$, then $E[\eta I_{[\eta > s+tx]}] = -E[\eta I_{[\eta < s+tx]}]$.

(ii) Since

$$\lim_{x \to -\infty} x^2 G(s_2 + tx) = 0$$

and

$$\lim_{x \to -\infty} \ln[(u_1 - u_2 x)/(v_1 - v_2 x)] = \ln(u_2/v_2),$$

$$\begin{aligned} &\lim_{x \to -\infty} x^2[-G(s_1 + tx)\ln(u_1 - u_2 x) + G(s_2 + tx)(\ln(v_1 - v_2 x) - c)] \\ &= \lim_{x \to -\infty} [-x^2(G(s_1 + tx) - G(s_2 + tx))\ln(u_1 - u_2 x) \\ &\quad - x^2 G(s_2 + tx)\ln[(u_1 - u_2 x)/(v_1 - v_2 x)] - cx^2 G(s_2 + tx)] \leqq 0. \end{aligned}$$

(iii) The proof of (iii) is similar to that of (ii).

Lemmas 3 and 4 make use of results of Tweedie (1976) for null recurrence and transience of Markov chains. Chan et al. (1985) show in their Lemma 2.1 that compact sets are what Tweedie calls status sets. This fact is used in the proof of Lemma 3.

Lemma 3. *Process* (1.1) *is null recurrent if* (2.6)–(2.10) *hold*.

Proof. Consider the function

$$g(x) = \begin{cases} \ln(\alpha + ax), & x > M > r_{l-1} \\ \ln(\beta - bx), & x < -M < r_1 \\ 0, & \text{otherwise,} \end{cases}$$

where a, b and M are positive constants and α and β are real numbers to be chosen for different regions (2.6)–(2.10). In what follows let $\varepsilon(j)$ be a generic random variable having the distribution of the $\{\varepsilon_t(j)\}$ sequence and whenever there is no ambiguity denote its distribution function by F. For convenience, let

$$k(x) = \phi(0, l) + \phi(1, l)x, \qquad h(x) = \phi(0, 1) + \phi(1, 1)x.$$

(i) Process (1.1) is null recurrent if (2.6) holds. First consider the region $\phi(1, l) = 1$, $\phi(0, l) = 0$, $0 \leqq \phi(1, 1) < 1$, and choose $a = b = \alpha = \beta = 1$. Suppose $x > M > r_{l-1}$, where M is to be chosen. Consider

$$E[\ln(\alpha + ak(x) + a\varepsilon(l))I_{[k(x)+\varepsilon(l)>M]}], \tag{3.1}$$

$$E[\ln(\beta - bk(x) - b\varepsilon(l))I_{[k(x)+\varepsilon(l)<-M]}], \tag{3.2}$$

$$(a/(\alpha + ak(x)))E[\varepsilon(l)I_{[\varepsilon(l)>M-k(x)]}], \tag{3.3}$$

$$(a^2/(2(\alpha + ak(x))^2))E[\varepsilon^2(l)I_{[M-k(x)<\varepsilon(l)<0]}], \tag{3.4}$$

$$(b/(\beta - bk(x)))E[\varepsilon(l)I_{[\varepsilon(l)<-M-k(x)]}]. \tag{3.5}$$

Since $E(\varepsilon^2(l)) < \infty$,

$$(3.4) = (a^2/(2(\alpha + ak(x))^2))E[\varepsilon^2(l)I_{[\varepsilon(l)<0]}] - o(x^{-2}),$$

and by Lemma 2.2(i), both (3.3) and (3.5) are $o(x^{-2})$.

For $x > M$, $\alpha + ak(x) > 0$, and thus by Lemma 1(i),

$$(3.1) \leqq (1 - F(M - k(x)))\ln(\alpha + ak(x)) + (3.3) - (3.4),$$

while $\beta - bk(x) < 0$, and thus by Lemma 1(ii),

$$(3.2) \leqq F(-M - k(x))(\ln(-\beta + bk(x)) - 2) + (3.5).$$

By Lemma 2(ii),

$$-F(M - k(x))\ln(\alpha + ak(x)) + F(-M - k(x))(\ln(-\beta + bk(x)) - 2) \leqq o(x^{-2}).$$

Since

$$E[g(X_t) \mid X_{t-1} = x] = (3.1) + (3.2),$$

we see that by choosing M large enough

$$\begin{aligned} E[g(X_t) \mid X_{t-1} = x] &\leqq g(x) - (a^2/(2(\alpha + ak(x))^2))E[\varepsilon^2(l)I_{[\varepsilon(l)<0]}] + o(x^{-2}) \\ &\leqq g(x), \quad \text{for } x > M. \end{aligned} \tag{3.6}$$

For $x < -M < r_1$ and $\phi(1, 1) = 0$, $E[g(X_t) \mid X_{t-1} = x]$ is a constant and is therefore less than $g(x)$.

For $x < -M < r_1$ and $0 < \phi(1, 1) < 1$, consider

$$E[\ln(\alpha + ah(x) + a\varepsilon(1))I_{[h(x)+\varepsilon(1)>M]}], \tag{3.7}$$

$$(3.8) \qquad E[\ln(\beta - bh(x) - b\varepsilon(1))I_{[h(x)+\varepsilon(1)<-M]}],$$

$$(3.9) \qquad (a/(\alpha + ah(x)))E[\varepsilon(1)I_{[\varepsilon(1)>M-h(x)]}],$$

$$(3.10) \qquad (b/(\beta - bh(x)))E[\varepsilon(1)I_{[\varepsilon(1)<-M-h(x)]}],$$

$$(3.11) \qquad (b^2/(2(\beta - bh(x))^2))E[\varepsilon^2(1)I_{[-M-h(x)>\varepsilon(1)>0]}].$$

Since $E(\varepsilon^2(1)) < \infty$

$$(3.11) = (b^2/(2(\beta - bh(x))^2))E[\varepsilon^2(1)I_{[\varepsilon(1)>0]}] - o(x^{-2}),$$

and by Lemma 2(i), both (3.9) and (3.10) are $o(x^{-2})$.

For $x < -M$, $\alpha + ah(x) < 0$, thus by Lemma 1(ii),

$$(3.7) \leqq (1 - F(M - h(x)))(\ln(-\alpha - ah(x)) - 2) - (3.9),$$

and $\beta - bh(x) > 0$, thus by Lemma 1(i),

$$(3.8) \leqq F(-M - h(x))\ln(\beta - bh(x)) - (3.10) - (3.11).$$

Now as

$$E[g(X_t) \mid X_{t-1} = x] = (3.7) + (3.8),$$

we see that by choosing M large enough $\beta - bh(x) \leqq \beta - bx$, and thus

$$\begin{aligned} F(-M - h(x))\ln(\beta - bh(x)) &\leqq F(-M - h(x))\ln(\beta - bx) \\ &= g(x) - (1 - F(-M - h(x)))\ln(\beta - bx). \end{aligned}$$

By Lemma 2(iii) then,

$$(1 - F(M - h(x)))(\ln(-\alpha - ah(x)) - 2) - (1 - F(-M - h(x)))\ln(\beta - bx) \leqq o(x^{-2}),$$

and thus

$$(3.12) \qquad \begin{aligned} E[g(X_t) \mid X_{t-1} = x] &\leqq g(x) - (b^2/(2(\beta - bh(x))^2))E[\varepsilon^2(1)I_{[\varepsilon(1)>0]}] + o(x^{-2}) \\ &\leqq g(x), \quad \text{for } x < -M. \end{aligned}$$

Next consider the region $\phi(1, l) = 1$, $\phi(0, l) = 0$, $\phi(1, 1) < 0$, and choose $b = -a\phi(1, l)$ and $\beta - \alpha = a\phi(0, 1)$. For $x > M > r_{l-1}$, (3.6) is obtained in a manner similar to the above. For $x < -M < r_1$, consider

$$(3.13) \qquad (a^2/(2(\alpha + ah(x))^2))E[\varepsilon^2(1)I_{[M-h(x)<\varepsilon(1)<0]}].$$

By Lemma 1

$$(3.7) \leqq (1 - F(M - h(x)))\ln(\alpha + ah(x)) + (3.9) - (3.13),$$

and

$$(3.8) \leqq F(-M - h(x))(\ln(-\beta + bh(x)) - 2) + (3.10).$$

From the choice of a, b, α and β, $\ln(\alpha + ah(x)) = \ln(\beta - bx) = g(x)$, and thus by Lemma 2(i) and (ii), for M large enough

$$E[g(X_t) \mid X_{t-1} = x] \leqq g(x) - (a^2/(2(\alpha + ah(x))^2))E[\varepsilon^2(1)I_{[\varepsilon(1)<0]}] + o(x^{-2})$$
(3.14)
$$\leqq g(x), \quad \text{for } x < -M.$$

(ii) Process (1.1) is null recurrent if (2.7) holds. By considering $Y_t = -X_t$, the result is obtained from (i).

(iii) Process (1.1) is null recurrent if (2.8) holds. Choose $a = b = \alpha = \beta = 1$. For $x > M > r_{l-1}$, (3.6) is obtained in a manner similar to the above. For $x < -M < r_1$, since $1 - h(x) \leqq 1 - x$,

$$F(-M - h(x))\ln(1 - h(x)) \leqq F(-M - h(x))\ln(1 - x).$$

From this, (3.12) is obtained in a manner similar to the above.

(iv) Process (1.1) is null recurrent if (2.9) holds. By considering $Y_t = -X_t$, the result is obtained from (iii).

(v) Process (1.1) is null recurrent if (2.10) holds. Choose $\beta - \alpha = b\phi(0, l) = a\phi(0, 1)$, $b = -a\phi(1, 1) = -a/\phi(1, l)$. For $x > M > r_{l-1}$, consider

$$(b^2/(2(\beta - bk(x))^2))E[\varepsilon^2(l)I_{[-M-k(x)>\varepsilon(l)>0]}]. \tag{3.15}$$

By Lemma 1

$$(3.1) \leqq (1 - F(M - k(x)))(\ln(-\alpha - ak(x)) - 2) - (3.3),$$

and

$$(3.2) \leqq F(-M - k(x))\ln(\beta - bk(x)) - (3.5) - (3.15).$$

From the choice of a, b, α and β

$$F(-M - k(x))\ln(\beta - bk(x)) = \ln(\alpha + ax) - (1 - F(-M - k(x)))\ln(\alpha + ax),$$

and thus by Lemma 2(i) and (iii), for M large enough

$$E[g(X_t) \mid X_{t-1} = x] \leqq g(x) - (b^2/(2(\beta - bk(x))^2))E[\varepsilon^2(l)I_{[\varepsilon(l)>0]}] + o(x^{-2})$$
(3.16)
$$\leqq g(x), \quad \text{for } x > M.$$

For $x < -M < r_1$, since $\ln(\alpha + ah(x)) = \ln(\beta - bx)$, (3.14) is obtained similarly.

It is obvious that the above test function g satisfies

$$g(x) > \sup_{|y| \leqq M} g(y), \quad \text{for } |x| > M,$$

and that the set $B_n = \{y : g(y) \leqq n\}$ is a compact set for all sufficiently large n and thus is a status set in Tweedie's (1976) sense. We may then apply Theorem 10.2 of Tweedie (1976) with set A taken to be $[-M, M]$, and the above test functions g to conclude the process is recurrent if any of (2.6)–(2.10) holds. As it has previously been shown by Chan et al. (1985) that the process is not ergodic if any of (2.6)–(2.10) holds, the process must be null recurrent on these regions.

Lemma 4. *Process* (1.1) *is transient if* (2.16) *holds*.

Proof. Let a and b be positive constants such that $-b/a = \phi(1, 1) = 1/\phi(1, l)$. Since $\phi(0, l) + \phi(1, l)\phi(0, 1) < 0$ we may, and do, choose α and β such that $-a\phi(0, 1) < a\alpha + b\beta < -b\phi(0, l)$. Choose c positive such that $c/a - \alpha > \max(0, r_{l-1})$ and $-c/b - \beta < \min(0, r_1)$. Consider the function

$$g(x) = \begin{cases} 1 - 1/a(x+\alpha), & x > c/a - \alpha \\ 1 - 1/c, & -c/b - \beta < x < c/a - \alpha \\ 1 + 1/b(x+\beta), & x < -c/b - \beta. \end{cases}$$

In what follows let $\varepsilon(j)$ be a generic random variable having the distribution of the $\{\varepsilon_t(j)\}$ sequence. Suppose $x > M > c/a - \alpha$, where M is to be chosen. Let

$$\gamma(x) = \phi(0, l) + \phi(1, l)x + \beta, \qquad \delta(x) = \phi(0, l) + \phi(1, l)x + \alpha.$$

Consider

$$-a^{-1}E[(1/(\delta(x) + \varepsilon(l)))I_{[\varepsilon(l) > c/a - \delta(x)]}], \tag{3.17}$$

$$-c^{-1}P(-c/b - \gamma(x) < \varepsilon(l) < c/a - \delta(x)), \tag{3.18}$$

$$1/a(x+\alpha) + b^{-1}E[(1/(\gamma(x) + \varepsilon(l)))I_{[\varepsilon(l) < -c/b - \gamma(x)]}]. \tag{3.19}$$

It is easy to show that both (3.17) and (3.18) are $o(x^{-2})$. Since

$$1/(\gamma(x) + \varepsilon(l)) = 1/\gamma(x) - \varepsilon(l)/\gamma(x)(\gamma(x) + \varepsilon(l)),$$

the second summand of (3.19) equals

$$F(-c/b - \gamma(x))/b\gamma(x) - E[(\varepsilon(l)/\gamma(x)(\gamma(x) + \varepsilon(l)))I_{[\varepsilon(l) < -c/b - \gamma(x)]}]$$

where F is the c.d.f. of $\varepsilon(l)$. Since

$$\begin{aligned} 1/(1 + \varepsilon(l)/\gamma(x)) &\leqq 1 + b\varepsilon(l)/c, && \text{for } 0 < \varepsilon(l) < -c/b - \gamma(x), \\ &\leqq 1, && \text{for } \varepsilon(l) \leqq 0, \end{aligned}$$

we have for x large enough

$$\begin{aligned} 0 &\geqq -x^2\varepsilon(l)/\gamma(x)(\gamma(x) + \varepsilon(l)) \\ &\geqq -x^2\varepsilon(l)(1 + b\varepsilon(l)/c)/\gamma^2(x) \\ &\geqq -2\varepsilon(l)(1 + b\varepsilon(l)/c)/\phi^2(1, l), \quad 0 < \varepsilon(l) < -c/b - \gamma(x), \end{aligned}$$

and

$$\begin{aligned} 0 &\leqq -x^2\varepsilon(l)/\gamma(x)(\gamma(x) + \varepsilon(l)) \\ &\leqq -x^2\varepsilon(l)/\gamma^2(x) \\ &\leqq -2\varepsilon(l)/\phi^2(1, l), \quad \varepsilon(l) \leqq 0. \end{aligned}$$

Thus, by the Lebesgue dominated convergence theorem,

$$\text{(3.20)} \quad \lim_{x\to\infty} x^2 E[-\varepsilon(l)/\gamma(x)(\gamma(x)+\varepsilon(l))I_{[\varepsilon(l)<-c/b-\gamma(x)]}] = E[-\varepsilon(l)/\phi^2(1,l)] = 0.$$

Thus, from (3.20) we see that (3.19) equals

$$\begin{aligned} &1/a(x+\alpha) + 1/b\gamma(x) - [1 - F(-c/b - \gamma(x))]/b\gamma(x) - o(x^{-2}) \\ &= (b\phi(0,l) + b\beta + a\alpha)/ab\gamma(x)(x+\alpha) - o(x^{-2}). \end{aligned}$$

Now as

$$E[g(X_t) \mid X_{t-1} = x] = g(x) + (3.17) + (3.18) + (3.19),$$

we see that by choosing M large enough

$$\begin{aligned} E[g(X_t) \mid X_{t-1} = x] &= g(x) + (b\phi(0,l) + b\beta + a\alpha)/ab\gamma(x)(x+\alpha) - o(x^{-2}) \\ &\geqq g(x), \qquad x > M. \end{aligned}$$

Similarly, for $x < -M < -c/b - \beta < r_1$, it can be shown that

$$E[g(X_t) \mid X_{t-1} = x] \geqq g(x).$$

We may thus apply Theorem 11.3 of Tweedie (1976) with the set A taken to be $[-M, M]$, and the above energy function g to conclude that the process is transient.

Lemma 5. *Process* (1.1) *is transient if* (2.13) *and* (2.14) *hold.*

Proof. Suppose (2.13) holds and begin the process at $x_0 < \min(0, r_1)$. Then until the first time the process exits $(-\infty, \min(0, r_1))$, it is a random walk with negative drift, and hence transient (see, e.g. Feller (1971), pp. 396–397). The proof of (2.14) is similar.

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