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A MULTIPLE-THRESHOLD AR(1) MODEL

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Abstract

We consider the model $Z_t = \phi(0,k) + \phi(1,k)Z_{t-1} + a_t(k)$ whenever $r_{k-1} < Z_{t-1} \leqq r_k$, $1 \leqq k \leqq l$, with $r_0 = -\infty$ and $r_l = \infty$. Here $\{\phi(i,k);\ i = 0,1;\ 1 \leqq k \leqq l\}$ is a sequence of real constants, not necessarily equal, and, for $1 \leqq k \leqq l$, $\{a_t(k), t \geqq 1\}$ is a sequence of i.i.d. random variables with mean 0 and with $\{a_t(k), t \geqq 1\}$ independent of $\{a_t(j), t \geqq 1\}$ for $j \neq k$. Necessary and sufficient conditions on the constants $\{\phi(i,k)\}$ are given for the stationarity of the process. Least squares estimators of the model parameters are derived and, under mild regularity conditions, are shown to be strongly consistent and asymptotically normal.

NON-LINEAR TIME SERIES; SETAR MODELS; AUTOREGRESSIVE MODELS; MARKOV CHAINS

1. Introduction

It seems generally agreed (see, for example, the discussion of Tong and Lim (1980)) that the class of threshold time series models forms one useful class of non-linear time series models. The practical relevance of non-linear analysis of time series data seems to be self-evident (see for example Tong (1983)).

Recently, Petruccelli and Woolford (1984) have discussed a simple first-order threshold model, which we denote by SETAR (2; 1, 1) following the usual convention in the area (see for example Tong (1983)). In fact they have considered the model

$$Z_t = \phi_1 Z_{t-1}^+ + \phi_2 Z_{t-1}^- + a_t, \qquad t = 1, 2, \cdots,$$

where $x^+ = \max(x, 0)$, $x^- = \min(x, 0)$ and $\{a_t\}$ is a white noise sequence. They

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have obtained the surprising result that for $\{Z_t\}$ to be ergodic it is both necessary and sufficient that

$$\phi_1 < 1, \quad \phi_2 < 1 \quad \text{and} \quad \phi_1\phi_2 < 1.$$

Thinking linearly, we could perhaps expect to require something like

$$|\phi_i| < 1, \qquad i = 1, 2,$$

which is in fact used in Tong (1983). Non-linearity seems to allow us greater freedom.

The present paper deals with the more general SETAR(l; $1, \cdots, 1$) model defined in Tong and Lim (1980). In particular, for any integer l, let

$$-\infty = r_0 < r_1 < \cdots < r_l = \infty$$

and define

$$Z_t = \phi(0,k) + \phi(1,k)Z_{t-1} + a_t(k) \quad \text{if } Z_{t-1} \in R_k, \tag{1.1}$$

where $R_k \equiv (r_{k-1}, r_k]$, $1 \leqq k \leqq l$. Equivalently, (1.1) may be written as

$$Z_t = \sum_{k=1}^{l} I(Z_{t-1} \in R_k)\{\phi(0,k) + \phi(1,k)Z_{t-1} + a_t(k)\}, \tag{1.2}$$

where $I(A)$ is the indicator function of the set A. In both (1.1) and (1.2) we take $\{\phi(i,k);\ i = 0, 1;\ 1 \leqq k \leqq l\}$ to be real constants and assume that for each k, $1 \leqq k \leqq l$, $\{a_t(k);\ t \geqq 1\}$ is a sequence of independent and identically distributed (i.i.d) random variables, each having a strictly positive density $f_k(\cdot)$, on $\mathbb{R}$, and mean 0. Additionally, we assume that $\{a_t(k)\}$ and $\{a_t(j)\}$ are independent for $j \neq k$.

In Section 2, we obtain necessary and sufficient conditions on the parameters $\{\phi(i,k);\ i = 0, 1;\ 1 \leqq k \leqq l\}$ for the process (1.2) to be ergodic. These conditions are broader than those given by Tong and Lim (1980) and more complex than those in Petruccelli and Woolford (1984).

In Section 3, assuming that $\{Z_t\}$ has a stationary distribution which has a second moment and that $\sigma^2(k) = E(a_t(k)^2)$ is finite, we establish the strong consistency of the least squares estimators for $\{\phi(i,k);\ i = 0, 1;\ 1 \leqq k \leqq l\}$ as well as for the estimator for $\sigma^2(k)$. In addition, a central limit theorem is shown to hold for the estimators for $\{\phi(i,k);\ i = 0, 1;\ 1 \leqq k \leqq l\}$. Finally, in Section 4, we give some concluding remarks.

2. Ergodicity

We note, as in Petruccelli and Woolford (1984), that $\{Z_t;\ t \geqq 0\}$, as defined in (1.2), is a Markov chain with state space $(\mathbb{R}, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-algebra on the real numbers $\mathbb{R}$. The transition density is given by

$$p(x,y)=\sum_{k=1}^{l} I(x\in R_k)f_k(y-\phi(0,k)-\phi(1,k)x). \tag{2.1}$$

Using the definitions in Orey (1971), we note also that $\{Z_t; t \geqq 0\}$ is μ-irreducible and aperiodic for μ taken to be Lebesgue measure on $\mathbb{R}$. However, contrary to the situation in Petruccelli and Woolford (1984), the transition law $\{P(x,\cdot)\}$, corresponding to (2.1), is not necessarily strongly continuous (a condition required to obtain ergodicity in Petruccelli and Woolford (1984)). Hence, we shall require the following lemma to prove the ergodicity results in Theorem 2.1 below.

Lemma 2.1. *Let* $\{P(x,\cdot)\}$ *be the transition law corresponding to the transition density* (2.1). *Then if* $\mathcal{K}$ *is the set of compact sets in* $\mathcal{B}$ *having positive Lebesgue measure, then* $0<\pi(K)<\infty$ *for all* $K\in\mathcal{K}$, *where* $\pi(\cdot)$ *is a subinvariant measure for* $\{Z_t\}$.

Proof. Let D be the set of discontinuities of $\{P(x,\cdot)\}$. Then, by construction, D is finite. By irreducibility we have a subinvariant measure $\pi(\cdot)$ such that

$$\pi(A)\geqq\int_{\mathbb{R}}\pi(dy)P(y,A),\qquad A\in\mathcal{B}.$$

Iterating the above equation we obtain

$$\beta(1-\beta)^{-1}\pi(A)\geqq\int_{\mathbb{R}}\pi(dy)G_\beta(y,A), \tag{2.2}$$

where $0<\beta<1$ and $G_\beta(y,A)=\sum_{n=1}^{\infty}\beta^nP^n(y,A)$. It is not hard to show that $G_\beta(y,\cdot)$ is continuous for $y\notin D$ and

$$\lim_{x\uparrow d}G_\beta(x,A)>0,\quad \lim_{x\downarrow d}G_\beta(x,A)>0$$

for all $d\in D$ whenever $\mu(A)>0$. Hence, we have, for any $K\in\mathcal{K}$,

$$\inf_{y\in K}G_\beta(y,A)>0$$

whenever $\mu(A)>0$. Using (2.2) and taking $A\in\mathcal{B}$ such that $0<\pi(A)<\infty$, then

$$\pi(K)\leqq\beta(1-\beta)^{-1}\pi(A)\left[\inf_{y\in K}G_\beta(y,A)\right]^{-1}<\infty.$$

That $\pi(K)>0$ follows from the fact that $\mu(K)>0$.

Remark. The above result is true for more general Markov chains than the one we have defined here. In particular, let $\{P(x,\cdot)\}$ be the transition law for an aperiodic and M-irreducible Markov chain with state space $(\mathbb{R},\mathcal{B})$. Let D be the finite set of discontinuities of $\{P(x,\cdot)\}$ and let $\pi(\cdot)$ be the associated subinvariant measure. Then, if $G_\beta(x,\cdot)=\sum_{n=1}^{\infty}\beta^n P^n(x,\cdot)$ and

$$\lim_{x\downarrow d} G_\beta(x,A)>0, \quad \lim_{x\uparrow d} G_\beta(x,A)>0$$

for all $d\in D$, whenever $M(A)>0$, we have that every compact $K\in\mathscr{B}$ has finite π-measure. This is an extension of Lemma 4.1 in Tweedie (1975) and indicates that the compact sets are still status sets in this case (see Tweedie (1976)).

We now prove necessary and sufficient conditions on the parameters $\{\phi(i,k);\ i=0,1;\ 1\leqq k\leqq l\}$ for the process $\{Z_t\}$ to be ergodic.

Theorem 2.1. *The process* $\{Z_t\}$, *defined by* (1.2), *is ergodic if and only if one of the following conditions holds*:

(2.3) $$\phi(1,1)<1, \quad \phi(1,l)<1, \quad \phi(1,1)\phi(1,l)<1;$$

(2.4) $$\phi(1,1)=1, \quad \phi(1,l)<1, \quad \phi(0,1)>0;$$

(2.5) $$\phi(1,1)<1, \quad \phi(1,l)=1, \quad \phi(0,l)<0;$$

(2.6) $$\phi(1,1)=1, \quad \phi(1,l)=1, \quad \phi(0,l)<0<\phi(0,1);$$

(2.7) $$\phi(1,1)\phi(1,l)=1, \quad \phi(1,1)<0, \quad \phi(0,l)+\phi(1,l)\phi(0,1)>0.$$

The proof of Theorem 2.1 is divided into two lemmas, the first of which proves sufficiency, the second necessity.

Lemma 2.2. *If* $\{Z_t\}$, *given by* (1.2), *satisfies one of* (2.3)–(2.7) *then* $\{Z_t\}$ *is ergodic.*

Proof. Similarly to the proof of Lemma 2.1 in Petruccelli and Woolford (1984), Lemma 2.1 above implies that the result of the theorem will follow from Theorem 3.1 of Tweedie (1975) if we can find a compact set $K\in\mathscr{B}$, having positive Lebesgue measure, and a non-negative measurable function g on $\mathbb{R}$ such that

(2.8) $$\int_{\mathbb{R}} p(x,y)g(y)dy\leqq g(x)-1, \qquad x\notin K$$

(2.9) $$\int_{\mathbb{R}} p(x,y)g(y)dy=\lambda(x)\leqq R<\infty, \qquad x\in K, \text{ for some fixed } R>0.$$

We prove that $\{Z_t\}$ is ergodic for each of (2.3)–(2.7) separately below by indicating a function g and a set K for which (2.8) and (2.9) hold.

(2.3): As in Petruccelli and Woolford (1984), we note the existence of positive constants a and b such that $1>\phi(1,1)>-(ba^{-1})$ and $1>\phi(1,l)>-(ab^{-1})$ and take

$$g(x)=\begin{cases} ax, & x>0 \\ b\,|x|, & x\leqq 0. \end{cases}$$

Then there is an $M>0$ such that (2.8) and (2.9) hold for $K=[-M,M]$.

(2.4): In this case we take

$$g(x)=\begin{cases} c_1x, & x>0 \\ -2[\phi(0,1)]^{-1}x, & x\leqq 0 \end{cases}$$

where $c_1>2|\phi(1,l)|[\phi(0,1)]^{-1}$. Then, again, there is an $M>0$ such that (2.8) and (2.9) hold for $K=[-M,M]$.

(2.5): By symmetry, we can take

$$g(x)=\begin{cases} -2[\phi(0,l)]^{-1}x, & x>0 \\ -c_2x, & x\leqq 0 \end{cases}$$

where $c_2>-2|\phi(1,1)|[\phi(0,l)]^{-1}$. The result follows as for (2.4).

(2.6): Again the result follows as for (2.4) and (2.5) with

$$g(x)=\begin{cases} 2[|\phi(0,l)|]^{-1}x, & x>0 \\ -2[\phi(0,1)]^{-1}x, & x\leqq 0. \end{cases}$$

(2.7): In this case we consider the Markov chain $\{Z_{2t}\}$ with transition law $\{P^2(x,\cdot)\}$. Taking

$$g(x)=\begin{cases} ax, & x>0 \\ -bx, & x\leqq 0, \end{cases}$$

where a and b are positive constants, we obtain, for $x\in R_j$,

$$\begin{aligned} I(x)&=\int_{\mathbf{R}} P^2(x,dy)g(y) \\ &=\sum_{k=1}^{l}\Big\{a\int_{\beta(k,x)}^{\infty}(u-\beta(k,x))\int_{R(k,j)} f_k(u-\phi(1,k)a)f_j(a)\,da\,du \\ &\qquad -b\int_{-\infty}^{\beta(k,x)}(u-\beta(k,x))\int_{R(k,j)} f_k(u-\phi(1,k)a)f_j(a)\,da\,du\Big\}, \end{aligned}$$

where

$$\beta(k,x)=-\phi(0,k)-\phi(1,k)\phi(0,j)-\phi(1,k)\phi(1,j)x$$

and

$$R(k,j)=\{y:y+\phi(0,j)+\phi(1,j)x\in R_k\}.$$

Clearly $I(x) \leqq B_j$, $0 < B_j < \infty$, for $x \in R_j$, $j = 2, \cdots, l-1$. In addition, it is not hard to show that there is an $M > 0$ such that

$$I(x) \leqq -bx - \frac{b}{2}(\phi(0,l) + \phi(1,l)\phi(0,1)), \qquad x < -M$$

$$I(x) \leqq ax + \frac{a}{2}(\phi(0,1) + \phi(1,1)\phi(0,l)), \qquad x > M.$$

Hence we can define a and b so that (2.8) and (2.9) hold for $K = [-M, M]$. Thus we conclude that $\{Z_{2t}\}$ is ergodic which, due to the irreducibility and aperiodicity of $\{Z_t\}$, implies the ergodicity of $\{Z_t\}$.

Lemma 2.3. *If* $\{Z_t\}$, *given by* (1.2), *does not satisfy one of* (2.3)–(2.7) *then* $\{Z_t\}$ *is not ergodic.*

Proof. We can distinguish four cases:
(i) $\phi(1,1) > 1$ or $\phi(1,l) > 1$.
(ii) ($\phi(1,1) = 1$ and $\phi(0,1) \leqq 0$) or ($\phi(1,l) = 1$ and $\phi(0,l) \geqq 0$).
(iii) $\phi(1,1) < 0$, $\phi(1,1)\phi(1,l) > 1$.
(iv) $\phi(1,1) < 0$, $\phi(1,1)\phi(1,l) = 1$ and $\phi(0,1)\phi(1,l) + \phi(0,l) \leqq 0$.

For Cases (i)–(iii), slight modification of the proof of Theorem 2.1 in Petruccelli and Woolford (1984) applies and we do not repeat the proof.

In order to prove Case (iv), we appeal to Theorem 9.1 (ii) in Tweedie (1976) to show that $\{Z_t\}$ is not ergodic (what Tweedie (1976) calls 'null'). Thus it suffices to find a non-negative Borel measurable function, $g(x)$, a set A of the form $[-a_1, a_2]$, $a_1 > 0$, $a_2 > 0$ and a constant $B > 0$ such that

$$\int_{\mathbb{R}} p(x,y)g(y)dy \geqq g(x), \qquad x \notin A \tag{2.10}$$

$$\int_{\mathbb{R}} p(x,y)|g(y) - g(x)| \leqq B, \qquad x \in \mathbb{R} \tag{2.11}$$

$$g(x) > \sup_{y \in A} g(y), \qquad x \notin A. \tag{2.12}$$

As $\phi(1,1)\phi(1,l) = 1$, there exist positive constants a and b such that

$$\phi(1,1) = -ba^{-1}, \quad \phi(1,l) = -ab^{-1}.$$

Define, for $\alpha, \beta > 0$, $k > 0$, $M > 0$,

$$g_{\alpha\beta}(x) = \begin{cases} ax + \alpha, & \text{if } x > 0 \\ -bx + \beta, & \text{otherwise} \end{cases}$$

and

$$I_{k,M}(x)=\begin{cases}k, & \text{if } |x|\leqq M\\ 0, & \text{otherwise,}\end{cases}$$

where the constants α, β, k and M are chosen so that

$$a\phi(0,1)\geqq\beta-\alpha\geqq b\phi(0,l) \tag{2.13}$$

$$M\geqq\max(|\phi(0,1)|,|\phi(0,l)|) \tag{2.14}$$

$$k\geqq(a+b)\max(|\phi(0,1)|,|\phi(0,l)|). \tag{2.15}$$

Note that (2.13) is always possible since $\phi(0,1)\phi(1,l)+\phi(0,l)\leqq 0$.

Thus, for

$$g(x)=g_{\alpha\beta}(x)+I_{k,M}(x)$$

and $x\in R_l$, large and positive,

$$\int_{\mathbf{R}}p(x,y)g(y)dy=\int_{\mathbf{R}}f_l(y-\phi(0,l)-\phi(1,l)x)g_{\alpha\beta}(y)dy$$
$$+\int_{\mathbf{R}}f_l(y-\phi(0,l)-\phi(1,l)x)I_{k,M}(y)dy.$$

But

$$\int_{\mathbf{R}}f_l(y-\phi(0,l)-\phi(1,l)x)g_{\alpha\beta}(y)dy$$
$$=\int_{\mathbf{R}}[-b(z+\phi(0,l)+\phi(1,l)x)+\beta]f_l(z)dz$$
$$+\int_{-\phi(0,l)-\phi(1,l)x}^{\infty}[(a+b)(z+\phi(0,l)+\phi(1,l)x)+\alpha-\beta]f_l(z)dz$$

so that

$$\int_{\mathbf{R}}p(x,y)g(y)dy=ax+\alpha+(\beta-\alpha)-b\phi(0,l)$$
$$+\int_0^{\infty}[(a+b)y+\alpha-\beta]f_l(y-\phi(0,l)-\phi(1,l)x)dy \tag{2.16}$$
$$+\int_{-M}^{M}kf_l(y-\phi(0,l)-\phi(1,l)x)dy.$$

Similarly, for $x\in R_1$, negative with $|x|$ large,

$$\int_{\mathbf{R}} p(x,y)g(y)dy = -bx + \beta + (\alpha - \beta) + a\phi(0,1)$$

$$-\int_{-\infty}^{0} [(a+b)y + \alpha - \beta] f_1(y - \phi(0,1) - \phi(1,1)x)dy \tag{2.17}$$

$$+\int_{-M}^{M} kf_1(y - \phi(0,1) - \phi(1,1)x)dy.$$

However, using (2.13)–(2.15) in (2.16) and (2.17), we see that, for $w > 0$ sufficiently large and $A = g^{-1}([0, w])$,

$$\int_{\mathbf{R}} p(x,y)g(y)dy \geqq g(x), \qquad x \notin A.$$

Clearly, (2.12) is satisfied and (2.11) can be easily shown to hold with

$$B \geqq aE(|a_1(1)|) + bE(|a_1(l)|) + 2(a|\phi(0,1)| + b|\phi(0,l)|)$$
$$+ 2|\beta - \alpha|.$$

Remarks. (1) We note that the conditions of ergodicity in Theorem 2.1 depend only on the parameters $\phi(0,1)$, $\phi(0,l)$, $\phi(1,1)$, $\phi(1,l)$ and, hence, only on the behavior of the process in regions R_1 and R_l.

(2) The regions of ergodicity are illustrated in Figures 2.1 a-c. We note that in the proof of Theorem 2.1 we have shown that the process is transient in regions (VI) and (VII). However, on those portions of regions (II)–(V) where the process is not ergodic we conjecture (but have been unable to prove) that the process is null recurrent.

Theorem 2.3. *Assume $E(|a_t(i)|^k) < \infty$, $1 \leqq i \leqq l$, and some integer k. Then, if $\phi(1,1)\phi(1,l) < 1$, $\Phi(1,1) < 1$, and $\phi(1,l) < 1$, the invariant probability distribution for the chain $\{Z_t\}$ has a finite kth moment, and the model is geometrically ergodic.*

Proof. Choose a, $b > 0$ such that $1 > \phi(1,1) > -(ba^{-1})$ and $1 > \phi(1,l) > -(ab^{-1})$. Let $c > 0$ and define

$$g(x) = \begin{cases} a^k x^k + c, & x > 0 \\ b^k |x|^k + c, & x \leqq 0. \end{cases}$$

It is not hard to show that for $|x|$ large

$$\int_{\mathbf{R}} p(x,y)g(y)dy \leqq (1-\varepsilon)g(x), \qquad \text{some } \varepsilon > 0.$$

The result then follows from Tweedie (1983).

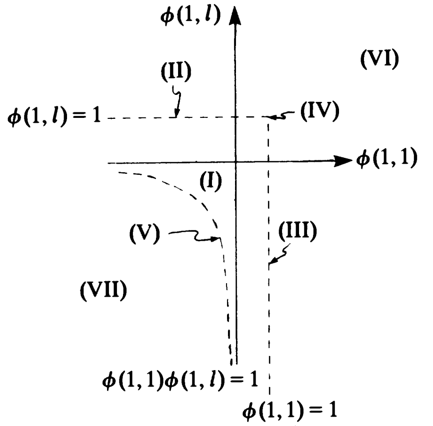
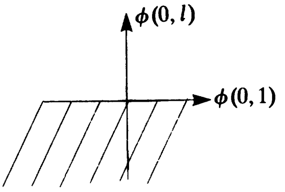
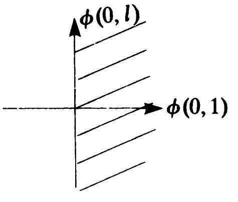


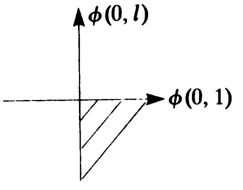
Figure 2.1a. $\{Z_t\}$ is ergodic for $\phi(1,1)$, $\phi(1,l)$ in (I) and not ergodic for $\phi(1,1)$, $\phi(1,l)$ in (VI) and (VII)



With $\phi(1,1)$, $\phi(1,l)$ in region (II) of Fig. 2.1a



With $\phi(1,1)$, $\phi(1,l)$ in region (III) of Fig. 2.1a



With $\phi(1,1)$, $\phi(1,l)$ in region (IV) of Fig. 2.1a

Figure 2.1b. $\{Z_t\}$ is ergodic for $\phi(0,1)$, $\phi(0,l)$ in the shaded regions

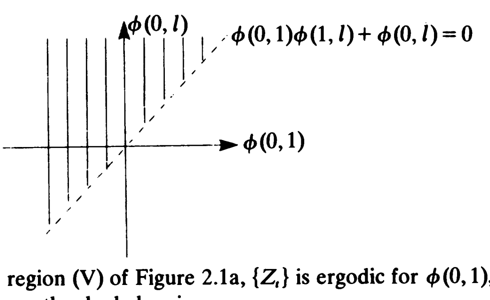


Figure 2.1c. With $\phi(1,1)$, $\phi(1,l)$ in region (V) of Figure 2.1a, $\{Z_t\}$ is ergodic for $\phi(0,1)$, $\phi(0,l)$ in the shaded region

Remark. These results strengthen those described in Theorem 2.2 and the remark following it in Petruccelli and Woolford (1984). However, it is not known if similar results hold under the other conditions of Theorem 2.1, each of which ensures the ergodicity of $\{Z_t\}$.

3. Estimation of model parameters

Throughout this section we make the following assumptions.

A1: $\{Z_t\}$ is ergodic and its stationary distribution has a finite second moment

and

A2: $$E(a_t(k)^2) = \sigma^2(k) < \infty, \qquad 1 \leqq k \leqq l.$$

We note that if $\phi(1,1)\phi(1,l) < 1$, $\phi(1,1) < 1$, $\phi(1,l) < 1$, and $\sigma^2(k) < \infty$, $1 \leqq k \leqq l$, then, by Theorem 2.3, the stationary distribution of $\{Z_t\}$ has a finite second moment. In what follows we take Z to be a random variable having as its distribution $\pi(\cdot)$, the invariant probability distribution for $\{Z_t\}$. $Z(k)$ will denote the random variable $ZI(Z \in R_k)$. We shall also denote by $J(k)$, $1 \leqq k \leqq l$, the set of integers $\{0 \leqq t \leqq n-1 : Z_t \in R_k\}$ and let $n(k)$ be the cardinality of $J(k)$.

Assuming r_k, $1 \leqq k \leqq l$, are known, the least squares estimators for the parameters $\{\phi(i,k)\}$ are given by (for $1 \leqq k \leqq l$ in all cases):

$$\hat{\phi}(1,k) = \left[\sum_{t \in J(k)} Z_t Z_{t+1} - \sum_{t \in J(k)} Z_t \sum_{t \in J(k)} Z_{t+1}/n(k) \right] \Big/ nS^2(k), \tag{3.1}$$

where

$$nS^2(k) = \sum_{t \in J(k)} Z_t^2 - \left(\sum_{t \in J(k)} Z_t \right)^2 \Big/ n(k),$$

$$\hat{\phi}(0,k) = n(k)^{-1} \left[\sum_{t \in J(k)} Z_{t+1} - \hat{\phi}(1,k) \sum_{t \in J(k)} Z_t \right], \tag{3.2}$$

and the corresponding natural estimator for $\sigma^2(k)$ is

$$\hat{\sigma}^2(k) = n(k)^{-1} \sum_{t \in J(k)} \{Z_{t+1} - \hat{\phi}(0,k) - \hat{\phi}(1,k) Z_t\}^2. \tag{3.3}$$

The next two theorems establish the strong consistency and asymptotic normality of the estimators in (3.1)–(3.3) when the process $\{Z_t\}$ is ergodic.

Theorem 3.1. *Under assumptions* A1 *and* A2, $\hat{\phi}(i,k)$ *and* $\hat{\sigma}^2(k)$, $i = 0, 1$; $1 \leqq k \leqq l$, *are strongly consistent estimators of* $\phi(i,k)$ *and* $\sigma^2(k)$, $i = 0, 1$; $1 \leqq k \leqq l$ *respectively.*

Proof. Rewrite (3.1), (3.2) as

$$\hat{\phi}(1,k)=\phi(1,k)+\left[\sum_{t\in J(k)} Z_t a_{t+1}(k)-\sum_{t\in J(k)} Z_t \sum_{t\in J(k)} a_{t+1}(k)/n(k)\right]\Big/ nS^2(k) \tag{3.4}$$

$$\begin{aligned}\hat{\phi}(0,k)&=\phi(0,k)\\ &+n(k)^{-1}\left[\sum_{t\in J(k)} Z_t^2 \sum_{t\in J(k)} a_{t+1}(k)-\sum_{t\in J(k)} Z_t \sum_{t\in J(k)} Z_t a_{t+1}(k)\right]\Big/ nS^2(k).\end{aligned} \tag{3.5}$$

By arguments analogous to those of Theorem 3.1 of Petruccelli and Woolford (1984) we have, as $n\to\infty$,

$$\begin{aligned} n(k)/n &\to \pi(R_k) \quad \text{a.s. } 1\leqq k\leqq l,\\ n^{-1}\sum_{t\in J(k)} Z_t^m &\to E(Z(k)^m) \quad \text{a.s. } m=1,2,\\ n^{-1}\sum_{t\in J(k)} Z_t a_{t+1}(k) &\to E(Z(k))E(a_1(k))=0 \quad \text{a.s.},\\ n^{-1}\sum_{t\in J(k)} a_{t+1}(k) &\to \pi(R_k)E(a_1(k))=0 \quad \text{a.s.},\end{aligned} \tag{3.6}$$

so that

$$S^2(k)\to E(Z(k)^2)-E^2(Z(k))/\pi(R_k).$$

Now by the Schwarz inequality $S^2(k)\geqq 0$ with equality holding if and only if $Z(k)$ is almost surely constant. As this is clearly not the case, $S^2(k)>0$. Applying (3.6) to (3.4) and (3.5) we see that, as $n\to\infty$,

$$\hat{\phi}(i,k)\to\phi(i,k) \quad \text{a.s., } i=1,2;\ 1\leqq k\leqq l.$$

To prove the strong consistency of $\sigma^2(k)$, rewrite (3.3) as

$$\begin{aligned}\hat{\sigma}^2(k)=n(k)^{-1}\sum_{t\in J(k)} \{&a_{t+1}^2(k)+(\hat{\phi}(0,k)-\phi(0,k))^2\\ &+(\hat{\phi}(1,k)-\phi(1,k))^2 Z_t^2-2a_{t+1}(k)(\hat{\phi}(0,k)-\phi(0,k))\\ &-2a_{t+1}(k)(\hat{\phi}(1,k)-\phi(1,k))Z_t\\ &+2(\hat{\phi}(0,k)-\phi(0,k))(\hat{\phi}(1,k)-\phi(1,k))Z_t\}.\end{aligned} \tag{3.7}$$

By applying (3.6) to (3.7) it is clear that

$$\hat{\sigma}^2(k)\to\sigma^2(k), \quad \text{a.s. as } n\to\infty.$$

To state the next theorem we shall need some notation. Let $\gamma(k)=E(Z(k)^2)-E^2(Z(k))/\pi(R_k)$, $1\leqq k\leqq l$. For $n=1,2,\cdots$ let $\mathbf{\Phi}(n)$ be the $2l\times 1$ vector whose $(2k-1)$th element is

$$[n\gamma(k)\pi(R_k)/\sigma^2(k)E(Z(k)^2)]^{1/2}(\hat{\phi}(0,k)-\phi(0,k))$$

and whose $2k$th element is

$$[n\gamma(k)/\sigma^2(k)]^{1/2}(\hat{\phi}(1,k)-\phi(1,k)).$$

For $k=1,\cdots,l$ let $D(k)$ be a 2×2 matrix with entries of 1 on the main diagonal and off-diagonal entries of $-E(Z(k))/[E(Z(k)^2)\pi(R_k)]^{1/2}$.

Let D be the $2l\times 2l$ matrix with matrices $D(k)$ along the main diagonal and entries of 0 everywhere else.

Theorem 3.2. *Under assumptions* A1 *and* A2, *as* $n\to\infty$, $\mathbf{\Phi}(n)$ *converges in distribution to an* $\mathrm{N}(\mathbf{0},D)$.

Proof. Consider $[n\gamma(k)\pi(R_k)/\sigma^2(k)E(Z(k)^2)]^{1/2}(\hat{\phi}(0,k)-\phi(0,k))$. This is easily shown to be asymptotically equivalent to

$$C_0(k,n)n^{-1/2}\sum_{t=1}^{n} Y_t(k)a_{t+1}(k),$$

where

$$C_0(k,n)=[\pi(R_k)/\gamma(k)\sigma^2(k)E(Z(k)^2)]^{1/2}$$

and

$$Y_t(k)=I_{R_k}(Z_t)[E(Z(k)^2)-E(Z(k))Z_t]/\pi(R_k).$$

In a similar manner we see that $[n\gamma(k)/\sigma^2(k)]^{1/2}(\hat{\phi}(1,k)-\phi(1,k))$ is asymptotically equivalent to

$$C_1(k,n)n^{-1/2}\sum_{t=1}^{n} W_t(k)a_{t+1}(k),$$

where

$$C_1(k,n)=[\gamma(k)\sigma^2(k)]^{-1/2}$$

and

$$W_t(k)=I_{R_k}(Z_t)[Z_t-E(Z(k))/\pi(R_k)].$$

Letting $\mathbf{\Psi}'=[\psi_1,\cdots,\psi_{2l}]$ be a $1\times 2l$ vector of real constants, it follows that for each n,

$$\mathbf{\Psi}'\mathbf{\Phi}(n)=n^{-1}\sum_{t=1}^{n}\sum_{k=1}^{l}[\psi_{2k-1}C_0(k,n)Y_t(k)+\psi_{2k}C_1(k,n)W_t(k)]a_{t+1}(k).$$

However,

$$\left\{\sum_{k=1}^{l}[\psi_{2k-1}C_0(k,n)Y_t(k)+\psi_{2k}C_1(k,n)W_t(k)]a_{t+1}(k),\ t\geqq 1\right\}$$

is a martingale difference sequence satisfying the conditions of Theorem 23.1 in Billingsley (1968). From this we can conclude that $\mathbf{\Psi}'\mathbf{\Phi}(n)$ converges in distribution to an $N(\mathbf{0}, \mathbf{\Psi}'D\mathbf{\Psi})$ which implies the result.

4. Discussion

The thresholds are assumed known in this paper. In practice, they are seldom known. The estimation of the thresholds remains a challenging problem. Although estimates have been proposed their sampling properties are unclear (see Tong (1983)).

The main results of this paper carry over quite easily to the fuzzy extension of model (1.1) as described by Tong (1983).

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