# AUTOREGRESSIVE PROCESSES WITH NORMAL STATIONARY DISTRIBUTIONS

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**Abstract.** For the strictly stationary AR($k$) process $Z_t = \lambda(\boldsymbol{Z}_{t-1}) + a_t$, with $\lambda : \mathbb{R}^k \to \mathbb{R}$, $\boldsymbol{Z}_{t-1} = [Z_{t-1}, Z_{t-2}, \ldots, Z_{t-k}]$ and $\{a_t\}$ an independent identically distributed white noise process, we partially characterize the $\lambda$ for which the stationary distribution of $Z_t$ is normal.

**Keywords.** AR processes; stationary distribution.

## 1. INTRODUCTION

Consider the strictly stationary AR($k$) process

$$Z_t = \lambda(\boldsymbol{Z}_{t-1}) + a_t \tag{1.1}$$

where $\boldsymbol{Z}_{t-1} = [Z_{t-1}, Z_{t-2}, \ldots, Z_{t-k}]$, the $\{a_t\}$ are independent identically distributed (i.i.d) zero-mean random variables and $\lambda : \mathbb{R}^k \to \mathbb{R}$ is a measurable function. In what follows we provide partial answers to the question: For which $\lambda$ does $Z_t$ have a normal stationary distribution?

We assume without loss of generality that the marginal stationary distribution of (1.1) is N(0, 1). Then by Cramer's theorem (Lukacs, 1960, p. 173) both $\lambda(\boldsymbol{Z}_{t-1})$ and $a_t$ must have normal distributions which we will take throughout to be N(0, $1 - \sigma^2$) and N(0, $\sigma^2$) respectively.

## 2. FIRST-ORDER PROCESSES

Throughout this section we assume a first-order model so that (1.1) becomes

$$Z_t = \lambda(Z_{t-1}) + a_t \tag{2.1}$$

Theorem 2.1. *The only differentiable $\lambda$'s for which the stationary distribution of (2.1) is normal are linear.*

Proof. Assume that $Z_{t-1} \sim$ N(0, 1), $\lambda(Z_{t-1}) \sim$ N(0, $1 - \sigma^2$). We first show that $\lambda$ is monotone. Suppose that it is not. Then $\lambda$ has a local maximum or minimum at some $z \in \mathbb{R}$, and $\lambda'(z) = 0$. If $M(\delta)$, $m(\delta)$ are the maximum and

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minimum of $\lambda$ on $[z-\delta, z+\delta]$, and if $\Phi(\cdot)$ is the N(0, 1) cumulative distribution function, we have

$$0 < 2\Phi'(z) = \lim_{\delta\to 0}\frac{\Phi(z+\delta)-\Phi(z-\delta)}{\delta}$$
$$\leqslant \lim_{\delta\to 0}\frac{P\{m(\delta)\leqslant \lambda(Z_{t-1})\leqslant M(\delta)\}}{\delta} = 0.$$

The last equality is obtained because

$$\lim_{\delta\to 0}\frac{M(\delta)-m(\delta)}{\delta} = 0.$$

Thus, by contradiction, $\lambda$ is monotone. Assume that it is strictly increasing. Then

$$\begin{aligned}\Phi(z) &= P\{\lambda(Z_{t-1}) \leqslant (1-\sigma^2)^{1/2}z\}\\ &= P[Z_{t-1}\leqslant \lambda^{-1}\{(1-\sigma^2)^{1/2}z\}]\\ &= \Phi[\lambda^{-1}\{(1-\sigma^2)^{1/2}z\}]\end{aligned}$$

which implies $\lambda(z) = (1-\sigma^2)^{1/2}z$. Similarly, $\lambda$ strictly decreasing implies $\lambda(z) = -(1-\sigma^2)^{1/2}z$. ■

However, Theorem 2.1 does not hold for merely continuous $\lambda$ as the following example shows. Assume that $a_t \sim \mathrm{N}(0, \sigma^2)$, some $0 < \sigma^2 < 1$ and define $\lambda$ on $[0, \infty)$ by

$$\lambda(z) = \begin{cases} 2(1-\sigma^2)^{1/2}z & (0\leqslant z\leqslant \frac{1}{2})\\ \zeta(z) & (\frac{1}{2} < z \leqslant \frac{3}{4})\\ (1-\sigma^2)^{1/2}(4z-3) & (\frac{3}{4} < z \leqslant 1)\\ (1-\sigma^2)^{1/2}z & (z > 1)\end{cases}$$

where

$$\zeta^{-1}(z) = \Phi^{-1}\left[\Phi\left\{\frac{z}{(1-\sigma^2)^{1/2}}\right\} - \Phi\left\{\frac{z}{4(1-\sigma^2)^{1/2}}-\frac{3}{4}\right\}\right.$$
$$\left. + \Phi\left\{-\frac{z}{2(1-\sigma^2)^{1/2}}\right\}\right] \qquad (-(1-\sigma^2)^{1/2}\leqslant z\leqslant 0).$$

Extend $\lambda$ to $(-\infty, 0)$ by skew symmetry, i.e. $\lambda(-z) = -\lambda(z)$, $z \geqslant 0$. Then $\lambda$ as described is continuous and, with this $\lambda$, (2.1) has an N(0, 1) stationary distribution.

THEOREM 2.2. *If $\lambda$ in (2.1) is $1-1$ and measurable and if $|\lambda(z)| = (1-\sigma^2)^{1/2}|z|$ almost everywhere (a.e.), then (2.1) has an* N(0, 1) *stationary distribution.*

PROOF. Since $\lambda$ is $1-1$ and $|\lambda(z)| = (1-\sigma^2)^{1/2}|z|$ a.e., $\lambda$ must be skew symmetric. If $A = \{z\in\mathbb{R}: \lambda(z) = (1-\sigma^2)^{1/2}z\}$, then $A$ is symmetric: $z\in A$ iff $-z\in A$.

Then if $Z_{t-1} \sim \mathrm{N}(0, 1)$,

$$P\{\lambda(Z_{t-1}) \leqslant z\} = P\left\{Z_{t-1} \leqslant \frac{z}{(1-\sigma^2)^{1/2}},\ Z_{t-1} \in A\right\} + P\left\{-Z_{t-1} \leqslant \frac{z}{(1-\sigma^2)^{1/2}},\ Z_{t-1} \notin A\right\} = P\left\{Z_{t-1} \leqslant \frac{z}{(1-\sigma^2)^{1/2}}\right\}$$

so that $\lambda(Z_{t-1}) \sim \mathrm{N}(0, 1-\sigma^2)$ which implies the result. ∎

Whether or not the converse of Theorem 2.2 is true is, as far as we know, an open question. However, if $\lambda$ leaves a suitable second measure invariant, then the converse holds. Specifically, if $\Phi(\cdot)$ is the N(0, 1) measure $\Phi(A) = \int_A \varphi(x)\, ds$, where $\varphi(x)$ is the N(0, 1) density, we set the following condition:

There is a measure $\mu$ equivalent to $\Phi$ on $\mathbb{R}$ with a Radon–Nikodym derivative $d\mu/d\Phi$ satisfying

(i) $(d\mu/d\Phi)(z_1) = (d\mu/d\Phi)(z_2)$ implies $|z_1| = |z_2|$ a.e. (C)
(ii) $\mu$ is invariant under the normalized transformation $\tau(\cdot) = \lambda(\cdot)/(1-\sigma^2)^{1/2}$.

THEOREM 2.3. *If $\lambda$ in (2.1) is $1-1$ and measurable with $\lambda^{-1}$ measurable and if condition (C) is satisfied, then process (2.1) has an* N(0, 1) *stationary distribution only if* $|\lambda(z)| = (1-\sigma^2)^{1/2}|z|$ *a.e.*

PROOF. Let $\tau(z) = \lambda(z)/(1-\sigma^2)^{1/2}$. As in Halmos (1956, p. 85), for any $A$ measurable,

$$\int_A \frac{d\mu}{d\Phi}\{\tau(z)\}\, d\Phi(z) = \mu\{\tau(A)\} = \mu(A) = \int_A \frac{d\mu}{d\Phi}(z)\, d\Phi(z)$$

so that

$$\frac{d\mu}{d\Phi}\{\tau(z)\} = \frac{d\mu}{d\Phi}(z) \text{ a.e.}$$

By assumption then $|\tau(z)| = |z|$ a.e. and the result follows. ∎

REMARK 2.1 Two measures $\mu$ which satisfy condition (C) are the Lebesgue measure and the $\mathrm{N}(0, \delta^2)$ measure, $\delta^2 \neq 1$. If $\mu$ is taken to be one of these a result obtained by Ghosh (1969) follows.

## 3. HIGHER-ORDER PROCESSES

In this section we assume that the process has the form (1.1) with $k > 1$. It is natural to ask, as we did in the previous section, for which $\lambda$ the stationary distribution of (1.1) is normal. However, since the marginal stationary distribution of $Z_t$ does not determine the joint distribution of $\boldsymbol{Z}_t$, there is little hope of characterizing $\lambda$ by this univariate distribution alone.

One natural generalization of the characterization of $\lambda$ by the normal marginal distribution in the first-order case is to require the $k$-dimensional stationary distribution of $\boldsymbol{Z}_t$ to be multivariate normal. Theorem 2.1 is then generalized by the following theorem.

THEOREM 3.1. *Suppose that $\lambda$ is continuous and that its ith partial derivative exists everywhere for some $1 \leqslant i \leqslant k$. If the stationary distribution of $\boldsymbol{Z}_t$ and of $[Z_t, Z_{t-1}, \ldots, Z_{t-i+2}, \lambda(\boldsymbol{Z}_t), Z_{t-i}, \ldots, Z_{t-k+1}]$ are both k-variate normal, then $\lambda$ is linear.*

PROOF. Let $\lambda_i(\boldsymbol{Z}_t) = [Z_t, Z_{t-1}, \ldots, Z_{t-i+2}, \lambda(\boldsymbol{Z}_t), Z_{t-i}, \ldots, Z_{t-k+1}]'$. Assume that $\boldsymbol{Z}_t \sim \mathrm{N}(\mathbf{0}, \Omega)$ and $\lambda_i(\boldsymbol{Z}_t) \sim \mathrm{N}(\mathbf{0}, \Sigma)$. Define $\boldsymbol{Z}_{t,i}$ to be the $(k-1)$-dimensional vector consisting of $\boldsymbol{Z}_t$ with its $i$th component removed. Let $\Sigma_i$ be the $(k-1) \times (k-1)$ matrix obtained from $\Sigma$ by deleting the $i$th row and column and similarly for $\Omega_i$. Let $\sigma_{ij}$ and $\omega_{ij}$ denote the $(i, j)$th entries in $\Sigma$ and $\Omega$ respectively and let

$$\boldsymbol{\sigma}_i = [\sigma_{i1}, \ldots, \sigma_{i,i-1}, \sigma_{i,i+1}, \ldots, \sigma_{ik}]',$$
$$\boldsymbol{\omega}_i = [\omega_{i1}, \ldots, \omega_{i,i-1}, \omega_{i,i+1}, \ldots, \omega_{ik}]'.$$

We then have that the distribution of

$$\tilde{\lambda}(\boldsymbol{Z}_t) = \frac{\lambda(\boldsymbol{Z}_t) - \boldsymbol{\sigma}_i' \Sigma_i^{-1} \boldsymbol{Z}_{t,i}}{(\sigma_{ii} - \boldsymbol{\sigma}_i' \Sigma_i^{-1} \boldsymbol{\sigma}_i)^{1/2}}$$

given $\boldsymbol{Z}_{t,i}$ is $\mathrm{N}(0, 1)$, as is the distribution of

$$\tilde{Z}_{t,i} = \frac{Z_{t-i+1} - \boldsymbol{\omega}_i' \Omega_i^{-1} \boldsymbol{Z}_{t,i}}{(\omega_{ii} - \boldsymbol{\omega}_i' \Omega_i^{-1} \boldsymbol{\omega}_i)^{1/2}}$$

also given $\boldsymbol{Z}_{t,i}$. It follows from the proof of Theorem 2.1 that, given $Z_{t,i}$,

$$\tilde{\lambda}(\boldsymbol{Z}_t) = \tilde{Z}_{t,i} \qquad \text{or} \qquad \tilde{\lambda}(\boldsymbol{Z}_t) = -\tilde{Z}_{t,i}.$$

Suppose that the former holds, without loss of generality. Then

$$\begin{aligned} \lambda(\boldsymbol{Z}_t) &= \delta_i(Z_{t-i+1} - \boldsymbol{\omega}_i' \Omega_i^{-1} \boldsymbol{Z}_{t,i}) + \boldsymbol{\sigma}_i' \boldsymbol{\Sigma}_i^{-1} Z_{t,i} \\ &= \delta_i Z_{t+i-1} + (\sigma_i' - \delta_i \omega_i') \boldsymbol{\Sigma}_i^{-1} Z_{t,i} \end{aligned} \tag{3.1}$$

where

$$\delta_i = \frac{(\sigma_{ii} - \boldsymbol{\sigma}_i' \Sigma_i^{-1} \sigma_i)^{1/2}}{(\omega_{ii} - \boldsymbol{\omega}_i' \Omega_i^{-1} \omega_i)^{1/2}}.$$

Because of continuity, it follows that $\lambda$ is a linear function of $\boldsymbol{Z}_t$. ∎

REMARK 3.1. If we asssume that the stationary distribution of process (1.1) is N(0, 1) and that of $\lambda(\boldsymbol{Z})$ is $N(0, 1-\sigma^2)$, then $\omega_{ii} = 1$, $\sigma_{ii} = 1 - \sigma^2$ and, since $\Omega_i = \Sigma_i$ and $\boldsymbol{\omega}_i = \boldsymbol{\sigma}_i$, (3.1) simplifies to

$$\delta_i Z_{t+i-1} + (1-\delta_i)\boldsymbol{\sigma}_i' \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Z}_{t,i}$$

where

$$\delta_i = \frac{(1 - \sigma^2 - \boldsymbol{\sigma}_i' \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\sigma}_i)^{1/2}}{(1 - \sigma_i' \Sigma_i^{-1} \sigma_i)^{1/2}}.$$

REMARK 3.2. Implicit in the proof of Theorem 3.1 is the following construction of nonlinear noncontinuous $\lambda$ which otherwise satisfy the conditions of the theorem. For any measurable $A$ in $\mathbb{R}^{k-1}$ define

$$\begin{aligned} \lambda(\boldsymbol{Z}_t) &= \delta_i Z_{t+i-1} + (\boldsymbol{\sigma}_i' - \delta_i \boldsymbol{\omega}_i')\Sigma_i^{-1} \boldsymbol{Z}_{t,i} \qquad (\boldsymbol{Z}_{t,i} \in A) \\ &= -\delta_i Z_{t+i-1} + (\boldsymbol{\sigma}_i' + \delta_i \boldsymbol{\omega}_i') \Sigma_i^{-1} \boldsymbol{Z}_{t,i} \qquad (\boldsymbol{Z}_{t,i} \notin A). \end{aligned}$$

Then if $\boldsymbol{Z}_t \sim N(\mathbf{0}, \Omega)$, $\boldsymbol{\lambda}_i(\boldsymbol{Z}_t) \sim N(\mathbf{0}, \Sigma)$.

Theorem 3.2 is an analog of Theorems 2.2 and 2.3 for the $k$th-order process (1.1), in which the measure $\mu$ of condition (C) is taken to be the Lebesgue measure, yielding the following condition:

the transformation $\tau: \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$\tau(\cdot) = \Omega^{1/2} \Sigma^{-1/2} \boldsymbol{\lambda}_i(\cdot) \tag{C'}$$

preserves Lebesgue measure.

Here $\boldsymbol{\lambda}_i$, $\Omega$ and $\Sigma$ are as defined in the proof of Theorem 3.1, and the powers $\frac{1}{2}$ and $-\frac{1}{2}$ signify the usual square root of a symmetric positive definite matrix and its inverse respectively.

THEOREM 3.2. *Suppose that $\lambda$ in (1.1) is such that for some $1 \leqslant i \leqslant k$ $\boldsymbol{\lambda}_i$ is 1 – 1 and measurable with $\boldsymbol{\lambda}_i^{-1}$ measurable, and that condition (C′) holds. Then the stationary distributions of $\boldsymbol{Z}_t$ and $\boldsymbol{\lambda}_i(\boldsymbol{Z}_t)$ are both k-variate normal if and only if*

$$\boldsymbol{\lambda}_i(z)' \Sigma^{-1} \boldsymbol{\lambda}_i(z) = z' \Omega^{-1} z \text{ a.e.}$$

PROOF. Necessity follows in essentially the same way as in the proof of Theorem 2.3. To show sufficiency we note that since $\tau$ preserves Lebesgue measure, so does

$$\tau^{-1}(z) = \lambda_i^{-1}(\Sigma^{1/2}\Omega^{-1/2}z).$$

Then if $Z_t \sim N(\mathbf{0}, \Omega)$ we have, for $A$ measurable in $\mathbb{R}^k$, $K = (2\pi)^{-k/2}|\Omega|^{-1/2}$,

$$\begin{aligned} P\{\tau(Z_t) \in A\} &= K \int_{\tau^{-1}(A)} \exp\left(-\frac{z'\Omega^{-1}z}{2}\right) dz \\ &= K \int_{\tau^{-1}(A)} \exp\left\{-\frac{\lambda_i(z)'\Sigma^{-1}\lambda_i(z)}{2}\right\} dz \\ &= K \int_A \exp\left(-\frac{z'\Omega^{-1}z}{2}\right) dz \\ &= P(Z_t \in A), \end{aligned}$$

so that $\tau(Z_t) \sim N(\mathbf{0}, \Omega)$. It follows that $\lambda_i(Z_t) \sim N(\mathbf{0}, \Sigma)$. ■

REMARK 3.3. In order for $\lambda_i$ to be $1-1$ and measurable with measurable inverse it suffices that $\lambda$ possesses these properties in its $i$th component, with the other $k-1$ components fixed.

REMARK 3.4. Eidlin (1971) has shown that algebraic and entire transformations of finite order from $\mathbb{R}^k$ to $\mathbb{R}^k$ that preserve normality must preserve equidensity contours of the distribution density. This implies the result of Theorem 2.1 for such $\lambda$. In addition, the equality at the end of the statement of Theorem 3.2 is precisely the preservation of equidensity contours of the distribution density of $\tau$.

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