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FULL-INFORMATION BEST-CHOICE PROBLEMS WITH RECALL OF OBSERVATIONS AND UNCERTAINTY OF SELECTION DEPENDING ON THE OBSERVATION

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Abstract

n i.i.d. random variables with known continuous distribution function F are observed sequentially with the object of choosing the largest. After any observation, say the kth, the observer may solicit any of the first k observations. If the $(k-t)$th is solicited, the probability of a successful solicitation may depend on t, the number of observations since the $(k-t)$th, and on the quantile of the $(k-t)$th observation. General properties of optimal selection procedures are obtained and the optimal procedures and their probabilities of success are derived in some special cases.

OPTIMAL STOPPING; SECRETARY PROBLEM; FULL INFORMATION

1. Introduction

The following best-choice problem was studied by Moser (1956), Guttman (1960) and Gilbert and Mosteller (1966): n i.i.d. random variables from a known continuous distribution F are observed sequentially with the object of choosing the largest. Neither past solicitation nor future knowledge of an observation is allowed and one choice must be made.

The present paper considers a similar problem in which past solicitation of an observation is allowed and such solicitation may be successful or unsuccessful. Further, the probability of a successful solicitation is allowed to depend on the number of observations from the solicited observation to the present and on the quantile of the solicited observation.

If, for example, the observations are scores on a competitive examination of applicants for a position and solicitation of an observation is a job offer to an applicant, then the probability that an applicant will accept an offer may depend on the time elapsed between the interview and the offer and on the applicant's exam score.

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This paper extends results of Yang (1974), Smith (1975) and Petruccelli (1981) in two ways. First, it considers a full-information problem—that is, F is known. Second, it allows the probability of a successful solicitation to depend on the observation.

Section 2 states assumptions and provides the basic formulas which are used to find an optimal procedure. Section 3 considers several special cases in detail, deriving the optimal procedure and the associated probabilities of success for each. Finally, Section 4 explores some general properties of optimal procedures.

2. The general solution

Let $X_1, \cdots, X_n$ be i.i.d. random variables from a known continuous distribution F. These random variables are observed sequentially with the object of choosing the largest.

Let $L_k = \max\{X_1, \cdots, X_k\}$ and call X_k a candidate if $XL_k = K_k$, $1 \leqq k \leqq n$.

For each $1 \leqq k \leqq n$ we denote the state of the process after observing X_k by a triple (x, k, t), $t < k$, if $x = L_k = X_{k-t}$ and if observation $k-t$ has not yet been solicited. If $L_k = x$ and the largest of the first k observations has been unsuccessfully solicited we describe the state of the process as (x, k, ∞).

If in state (x, k, t) we solicit the largest among the first k observations (X_{k-t} if $t < \infty$), the probability that we obtain this observation is $q(F(x), t)$. We assume:

(i) $q(u, \infty) \equiv 0$, $0 \leqq u \leqq 1$. That is, only one solicitation is allowed for each observation.

(ii) $q(u, t)$ is non-increasing in u for fixed t and in t for fixed u.

This last seems reasonable in light of the example of the last section: if the observations are scores on a competitive examination for a position then Condition (ii) states that the probability of a particular applicant accepting an offer of employment decreases as the time between the interview and offer increases and that the more competitive is the applicant the lower is the probability that he will accept an offer at any given time.

We shall assume that the observations are from a $U[0, 1]$ distribution as no generality is lost in doing so.

Suppose the process is in state (x, k, t). We then have two options: we may solicit the best among the first k observations or we may elect to observe the $(k+1)$th observation. Let $\delta_b(x, k, t)$ be the probability of choosing the largest in the first instance assuming we continue in an optimal manner. Let $\delta_f(x, k, t)$ be the same probability in the second instance again assuming optimal behavior for the future. Then clearly an optimal procedure will, in state (x, k, t), solicit the largest of the first k observations if and only if $\delta_b(x, k, t)$ exceeds $\delta_f(x, k, t)$.

Let $\delta(x, k, t) = \max\{\delta_b(x, k, t),\ \delta_f(x, k, t)\}$.

The following recursive formulas are easily derived:

$$\delta_f(x, k, t) = \int_x^1 \delta(y, k+1, 0)\, dy + x\delta(x, k+1, t+1) \tag{2.1}$$

$$\delta_b(x, k, t) = x^{n-k} q(x, t) + (1 - q(x, t))\delta(x, k, \infty) \tag{2.2}$$

$$\delta(x, k, \infty) = \int_x^1 \delta(y, k+1, 0)\, dy + x\delta(x, k+1, \infty) \tag{2.3}$$

$$\delta(x, n, t) = q(x, t). \tag{2.4}$$

(2.3) and (2.4) imply

$$\delta(x, k, \infty) = \sum_{j=0}^{n-k-1} x^j \int_x^1 \delta(y, k+j+1, 0)\, dy. \tag{2.5}$$

The following notation will be used throughout: $I_A(\cdot)$ will be the usual indicator function of the set A. $I(E)$ where E is an event, will be another kind of indicator function, taking the value 1 if E occurs, 0 if not.

Vacuous sums will take value 0; vacuous products value 1.

For observations $X_1, \cdots, X_k$, $L_k = \max\{X_1, \cdots, X_k\}$; $L_0 = 0$.

$[\cdot]$ is the greatest integer function.

3. Special cases

In this section we find, for several choices of $q(\cdot, \cdot)$ an optimal procedure and the probability of choosing the largest observation using this procedure. Asymptotic results are also obtained.

3.1. *Case* 1. $q(x, t) = q(x)$. Intuitively when the probability of a successful solicitation remains the same over time an optimal procedure will always observe the next observation since nothing is lost in doing so. The result is a procedure which observes all n X's before making an offer to the largest. This intuitive argument can be made rigorous by using (2.1)–(2.5) to show that $\delta_f(x, k, t)$ exceeds $\delta_b(x, k, t)$ for all $x \in [0, 1]$, $1 \leqq k < n$, $t < k$.

The probability of choosing the largest is then easily seen to be

$$n\int_0^1 q(x) x^{n-1}\, dx,$$

and if $\lim_{x\to 1} q(x) = q$, the asymptotic probability, as $n \to \infty$, of choosing the largest is q.

3.2. *Case* 2. $q(x, 0) = q(x)$; $q(x, t) = p(x)$, $t \geqq 1$; $q(x) \geqq p(x)$, $x \in [0, 1]$.

Theorem 3.1. For Case 2:

(i) *Form of the optimal rule.* If $p(x)/q(x)$ is non-increasing in x then for each n there are numbers $\{d(j,n)\}_{j=1}^{n}$ with $d(j,n)=d(j,n;q(\cdot),p(\cdot))$ and $d(j,n)$ decreasing in j, such that an optimal procedure is to solicit the first candidate X_k which exceeds $d(k,n)$, $1 \leqq k \leqq n$. If this solicitation is unsuccessful then each candidate that appears is solicited until a successful solicitation is obtained or until no observations remain.

(ii) *Formula for decision numbers $d(k,n)$.* $d=d(k,n)$ satisfies

$$(3.1) \qquad (q(d)-p(d))/q(d)=\sum_{\nu=1}^{n-k}\binom{n-k}{\nu}(1-d)^{\nu}d^{-\nu}g(n,\nu,d)$$

where $g(n,\nu,d)$ is a function defined in the proof below.

(iii) *Probability of successful solicitation.* The probability of choosing the largest observation using the optimal procedure is

$$(3.2) \qquad P(n)=\sum_{k=1}^{n}\{P_1(k,n)+P_2(k,n)+P_3(k,n)\}$$

where

$$\begin{aligned}
P_1(k,n)&=(k-1)\left\{\sum_{\nu=1}^{n-k}\binom{n-k}{\nu}\int_{d(k,n)}^{d(k-1,n)}(1-x)^{\nu}x^{n-\nu-1}g(n,\nu,x)\,dx\right.\\
&\qquad\left.+\int_{d(k,n)}^{d(k-1,n)}p(x)x^{n-1}\,dx\right\}, \qquad k\geqq 2\\
&=0, \qquad\qquad\qquad\qquad\qquad\qquad k=1.\\
P_2(k,n)&=d^{k-1}(k,n)\left\{\int_{d(k,n)}^{1}q(x)x^{n-k}\,dx\right.\\
&\qquad\left.+\sum_{\nu=1}^{n-k}\binom{n-k}{\nu}\int_{d(k,n)}^{1}(1-q(x))(1-x)^{\nu}x^{n-k-\nu}g(n,\nu,x)\,dx\right\}\\
P_3(k,n)&=(k-1)\left\{\int_{d(k,n)}^{d(k-1,n)}y^{k-2}\int_{y}^{1}q(x)x^{n-k}\,dx\,dy\right.\\
&\qquad+\sum_{\nu=1}^{n-k}\binom{n-k}{\nu}\int_{d(k,n)}^{d(k-1,n)}y^{k-2}\int_{y}^{1}(1-q(x))x^{n-k-\nu}(1-x)^{\nu}\\
&\qquad\left.\times g(n,\nu,x)\,dx\,dy\right\}, \qquad k\geqq 2\\
&=0, \qquad\qquad\qquad\qquad\qquad\qquad k=1.
\end{aligned}$$

(iv) *Asymptotic formula for decision numbers.* If $\lim_{x\to 1}p(x)=p$, $\lim_{x\to 1}q(x)=q$, then there exist $\{b_j\}_{j=1}^{\infty}$ such that $d(k,n)=1-b_{n-k}/(n-k)$. In addition $\lim_{j\to\infty} b_j = b$ where b satisfies

$$(q-p)/q^2 = \sum_{\nu=1}^{\infty} \prod_{j=2}^{\nu} (1-q/j)b^\nu/\nu! \tag{3.3}$$

(v) *Asymptotic probability of successful solicitation.* Under the assumptions in (iv) the asymptotic probability as $n\to\infty$ of choosing the largest observation is

$$\begin{aligned}&\left[p+q\sum_{\nu=1}^{\infty}\prod_{j=2}^{\nu}(1-q/j)b^\nu/\nu!\right]\left[e^{-b}-b\int_1^\infty (e^{-bv}/v)\,dv\right]\\ &+q(e^b-1)\int_1^\infty (e^{-bv}/v)\,dv\\ &+\left[q(1-q)\sum_{\nu=1}^{\infty}(\nu!)^{-1}\prod_{j=2}^{\nu}(1-q/j)\int_0^b u^\nu e^{-u}\,du\right]\left[e^b\int_1^\infty (e^{-bv}/v)\,dv\right].\end{aligned} \tag{3.4}$$

Remarks.

(i) Intuitively the optimal rule may be thought of as follows: if an observation is a candidate and is 'good enough' (i.e. exceeds its decision number $d(k, n)$) then it will be solicited. As more observations are taken 'good enough' becomes a less stringent hurdle and a less important criterion compared with the requirement that the observation be a candidate. Notice also that even if L_k exceeds $d(k, n)$ and has not yet been solicited, it will not be solicited at time k unless $X_k = L_k$. This is because, as in Case 1, $p(x)$, the probability of a successful solicitation of that observation, does not decrease with time.

(ii) The condition that $p(x)/q(x)$ be non-increasing in x can be interpreted as saying that the relative loss in the probability of successful solicitation by not soliciting immediately is greater the larger the observation is. Thus, for example, the better the applicant for the job, the less is the relative probability of his acceptance of an offer 'later' compared to his probability of accepting an immediate offer. This is reasonable if one believes that better applicants are chosen faster by the market than are lesser applicants.

(iii) If $q(x)\equiv q$, $p(x)\equiv p$ then

$$g(n, \nu, x) = q\prod_{j=2}^{\nu}(1-q/j).$$

This results in (3.1) and (3.2) taking a form which is computationally much simpler. The asymptotic results (3.3) and (3.4) are unchanged.

Proof of Theorem 3.1. By using Equations (2.1)–(2.5) and an induction argument we can show that

$$\delta_f(x, l, t) > \delta_b(x, l, t) \qquad 0 < t < l \leqq n-1.$$

This implies

$$\delta_b(x, k, 0) = q(x)x^{n-k} + (1-q(x))\delta(x, k, \infty)$$
$$\delta_f(x, k, 0) = p(x)x^{n-k} + \delta(x, k, \infty)$$

so that $\delta_f(x, k, 0) \geqq \delta_b(x, k, 0)$ iff

$$x^{k-n}\delta(x, k, \infty) \geqq 1 - p(x)/q(x). \tag{3.5}$$

By assumption the right side of (3.5) is non-decreasing while the left side is decreasing in x. Thus there is a $d(k, n) = d(k, n; q(\cdot), p(\cdot))$ such that (3.5) holds iff $x \leqq d(k, n)$.

Assume $x \leqq d(k, n)$. Then from (3.5) we obtain

$$\begin{aligned} x^{n-k+1}(1-p(x)/q(x)) &\leqq \sum_{j=1}^{n-k} x^j \int_x^1 \delta(y, k+j, 0)\, dy \\ &< \delta(x, k-1, \infty) \end{aligned}$$

which implies

$$\delta_f(x, k-1, 0) > \delta_b(x, k-1, 0)$$

and thus $x < d(k-1, n)$. Hence $d(k, n)$ is decreasing in k. This proves *i*.

Let $\nu(n, k, x) = \sum_{j=k+1}^{n} I_{(x,1]}(X_j)$; that is, the number of $X_{k+1}, \cdots . X_n$ which exceed x. $\nu(n, k, x)$ has a binomial distribution with parameters $n-k$ and $1-x$.

Let $c(\nu) = c(\nu(n, k, x))$ be the number of candidates among the ν which exceed x; that is, the number which at the time of observation are the largest of all so far observed. $c(\nu)$ is distributed as the sum $\sum_{i=1}^{\nu} Y_i$ where ν and the Y_i are mutually independent and $P(Y_i = 1) = i^{-1} = 1 - P(Y_i = 0)$.

Let $1 > Z_1 > Z_2 > \cdots > Z_{c(\nu)} > x$ denote the scores (X-values) of the $c(\nu)$ candidates which exceed x.

Then, for $\nu \geqq 1$ define

$$g(n, \nu, x) = E\left(q(Z_1) \prod_{j=2}^{c(\nu)} (1-q(Z_j)) \;\middle|\; \nu, X_k = L_k = x\right).$$

Notice that $g(n, \nu, x)$ is the probability, given ν and $X_k = L_k = x$, of choosing the largest observation using the strategy of immediately soliciting each candidate among $X_{k+1}, \cdots, X_n$ that appears. A computational formula for $g(n, \nu, x)$ is found in Appendix A.

Let $\sigma(k, n)$ denote the strategy of not soliciting X_k but of immediately soliciting each candidate among $X_{k+1}, \cdots, X_n$ that appears. Let $\gamma(k, n)$ denote the strategy of soliciting X_k and if rejected of continuing as does $\sigma(k, n)$. Then

$$\begin{aligned} &P(X_{\sigma(k,n)} = L_n \mid X_k = L_k = x) \\ &\quad = \sum_{\nu=1}^{n-k} \binom{n-k}{\nu}(1-x^{\nu})x^{n-k-\nu} g(n, k, x) + p(x)x^{n-k}. \end{aligned} \tag{3.6}$$

$$P(X_{\gamma(k,n)}=L_n \mid X_k=L_k=x)$$
$$=(1-q(x))\sum_{\nu=1}^{n-k}\binom{n-k}{\nu}(1-x)^\nu x^{n-k-\nu}g(n,k,x)+q(x)x^{n-k}. \tag{3.7}$$

$d(k,n)$ is the value of x which equates (3.6) and (3.7). This proves (3.1).

Let $\tau(n)$ denote the optimal procedure. That is, $\tau(n)=k$ if and only if the optimal procedure chooses X_k. Let

$$\begin{aligned}
P_1(k,n)&=P(\{X_{\tau(n)}=L_n\}\cap\{d(k,n)<L_k=L_{k-1}\leqq d(k-1,n)\})\\
&=\int_{d(k,n)}^{d(k-1,n)}P(X_{\sigma(k,n)}=L_n \mid X_k=L_k=x)\\
&\quad\times P(L_k=L_{k-1}=x)\,dx \qquad 2\leqq k\leqq n,\\
P_2(k,n)&=P(\{X_{\tau(n)}=L_n\}\cap\{L_{k-1}\leqq d(k,n)<X_k\})\\
&=\int_{d(k,n)}^{1}\int_0^{d(k,n)}P(X_{\gamma(k,n)}=L_n \mid X_k=L_k=x)\\
&\quad\times P(L_{k-1}=y, X_k=x)\,dy\,dx \qquad 1\leqq k\leqq n,\\
P_3(k,n)&=P(\{X_{\tau(n)}=L_n\}\cap\{d(k,n)<L_{k-1}\leqq d(k-1,n), X_k>L_{k-1}\})\\
&=\int_{d(k,n)}^{d(k-1,n)}\int_y^1 P(X_{\gamma(k,n)}=L_n \mid X_k=L_k=x)\\
&\quad\times P(L_{k-1}=y, X_k=x)\,dy\,dx, \qquad 2\leqq k\leqq n.
\end{aligned}$$

With some computation these expressions yield those in (iii). (3.2) follows easily if we note that $P_1(k,n)+P_2(k,n)+P_3(k,n)$ is the probability that $X_{\tau(n)}=L_n$ and that solicitation is first allowed at time k, $k\geqq 2$. $P_2(1,n)$ is the same probability at time 1.

Since $q(x), p(x)\geqq 0$ and are non-increasing $\lim_{x\to 1} q(x)=q$, $\lim_{x\to 1} p(x)=p$ exist.

Clearly (see (3.1) and Appendix A) the quantity $j(1-d(n-j,n))$ is a function only of j (not of $n-j$) and therefore we can write $d(k,n)=1-b_{n-k}/(n-k)$. Let $b^*=\limsup_{j\to\infty} b_j$, $b_*=\liminf_{j\to\infty} b_j$. By using (3.1) and bounds on $g(n,\nu,x)$ it can be shown that $0\leqq b_*\leqq b^*<(q-p)/q^2$. Let $b_{n-\alpha(n)}$ and $b_{n-\beta(n)}$ be subsequences converging to b^* and b_* respectively. (3.1) can be written as

$$d^{n-k}(k,n)(q(d(k,n))-p(d(k,n)))/q(d(k,n))$$
$$=E(I_{(0,\infty)}(\nu(n,k,d(k,n)))q(Z_1)\prod_{j=2}^{c(\nu)}(1-q(Z_j)) \mid X_k=L_k=d(k,n)).$$

Let $k=\alpha(n)$ and let $n\to\infty$. Then we obtain

$$\begin{aligned} e^{-b^*}(q-p)/q &= qE(I_{(0,\infty)}(N^*)(1-q)^{\sum_{i=1}^{N^*} Y_i}) \\ &= qE(I_{(0,\infty)}(N^*)\prod_{j=1}^{N^*}(1-q/j)) \end{aligned}$$

where N^* has a Poisson distribution with parameter b^*. By similarly letting $k=\beta(n)$ and letting $n\to\infty$ we obtain

$$e^{-b_*}(q-p)/q = qE\Big(I_{(0,\infty)}(N_*)\prod_{j=1}^{N_*}(1-q/j)\Big)$$

where N_* has a Poisson distribution with parameter b_*.

These two expressions imply that $b_*=b^*=b$ and that b satisfies (3.3).

To prove (3.4) we will view the problem from a different perspective. Let $c(n)$ be the number of candidates among $X_1,\cdots,X_n$, and define

$$\begin{aligned} \xi_1(n) &= kI(X_k=L_n) \\ \xi_j(n) &= kI(X_k=L_{\xi_{j-1}(n)-1}), \qquad c(n)\geqq j\geqq 2 \\ &= 0 \qquad j>c(n) \end{aligned}$$

so that $\xi_j(n)$ is the arrival time of the jth-largest candidate among $X_1,\cdots,X_n$, $j\geqq 1$.

For each $j\geqq 2$, define

$$\begin{aligned} V_1(n) &= \xi_1(n)/n & Z_1(n) &= n(1-L_{\xi_1(n)}) \\ V_j(n) &= \xi_j(n)/\xi_{j-1}(n) & Z_j(n) &= \xi_{j-1}(n)(1-L_{\xi_j(n)}/L_{\xi_{j-1}(n)}). \end{aligned}$$

For each j, since $c(n)\to\infty$ as $n\to\infty$ a.s.,

$$(V_1(n),\cdots,V_j(n),Z_1(n),\cdots,Z_j(n)) \xrightarrow[n\to\infty]{\mathscr{D}} (V_1,\cdots,V_j,Z_1,\cdots,Z_j)$$

where the V's and Z's are mutually independent, $V_i\sim U[0,1]$, $Z_i\sim\exp(1)$.

$$\begin{aligned} P(X_{\tau(n)}=L_n) &= \sum_{j=2}^{n} P(X_{\tau(n)}=L_n, L_{\xi_j(n)}\leqq d(\xi_j(n),n), L_{\xi_{j-1}(n)}>d(\xi_{j-1}(n),n)) \\ &\quad + P(X_{\tau(n)}=L_n, L_{\xi_1(n)}<d(\xi_1(n),n)) \\ &= \sum_{j=2}^{n} E(q(L_{\xi_1(n)})\prod_{l=2}^{j-1}(1-q(L_{\xi_l(n)})) \\ &\quad \times I(L_{\xi_j(n)}\leqq d(\xi_j(n),n), L_{\xi_{j-1}(n)}>d(\xi_{j-1}(n),n))) \\ &\quad + E(p(L_{\xi_1(n)})I(L_{\xi_1(n)}<d(\xi_1(n),n)) \\ &\to \sum_{j=2}^{\infty} q(1-q)^{j-2}\Lambda_j + p\Lambda_1 \end{aligned}$$

where

$$\Lambda_j = P(b/(1-V_1\cdot\cdots\cdot V_j) < Z_1 + Z_2/V_1 + \cdots + Z_j/V_1\cdot\cdots\cdot V_{j-1},$$
$$b/(1-V_1\cdot\cdots\cdot V_{j-1}) > Z_1 + Z_2/V_1 + \cdots + Z_{j-1}/V_1\cdot\cdots\cdot V_{j-2})$$
$$\Lambda_1 = P(Z_1 > b/(1-V_1)).$$

This shows that we may assume that $d(k, n) = 1 - b/(n-k)$ and that $q(x) \equiv q$ and $p(x) \equiv p$ for the purpose of finding $\lim_{n\to\infty} P(X_{\tau(n)} = L_n)$.

Under these assumptions $g(n, \nu, x) = q\prod_{j=2}^{\nu}(1-q/j)$ and therefore

$$P_1(k, n) = (k-1)\left\{q\sum_{\nu=1}^{n-k}\binom{n-k}{\nu}\prod_{j=2}^{\nu}(1-q/j)\right.$$
$$\left.\times\int_{d(k,n)}^{d(k-1,n)} x^{n-\nu-1}(1-x)^{\nu}\,dx + p(d^n(k-1, n) - d^n(k, n))/n\right\}.$$

Let an denote $[an]$ where $0 < a < 1$. Then

$$\begin{aligned} nP_1(an, n) &= (an-1)p((1-b/(n-an+1))^n - (1-b/(n-an))^n) \\ &\quad + n(an-1)q\sum_{\nu=1}^{n-an}\binom{n-an}{\nu}\prod_{j=2}^{\nu}(1-q/j) \\ &\quad \times\int_{1-b/(n-an)}^{1-b/(n-an+1)} x^{n-\nu-1}(1-x)^{\nu}\,dx. \end{aligned} \tag{3.8}$$

The first summand of (3.8) converges to $pabe^{-b/(1-a)}/(1-a)^2$. The second converges to

$$\left[qabe^{-b/(1-a)}\sum_{\nu=1}^{\infty}\prod_{j=2}^{\nu}(1-q/j)b^{\nu}/\nu!\right]\Big/(1-a)^2.$$

Similarly

$$nP_2(an, n) \to q(1-e^{-b})e^{-ab/(1-a)}/(1-a)$$
$$+\left[q(1-q)e^{-ab/(1-a)}\sum_{\nu=1}^{\infty}(\nu!)^{-1}\prod_{j=2}^{\nu}(1-q/j)\int_0^b u^{\nu}e^{-u}\,du\right]\Big/(1-a),$$

and $nP_3(an, n) \to 0$. Thus $P(X_{\tau(n)} = L_n)$ converges to

$$\left[p + q\sum_{\nu=1}^{\infty}\prod_{j=2}^{\nu}(1-q/j)b^{\nu}/\nu!\right]b\int_0^1 (ae^{-b/(1-a)}/(1-a)^2)\,da$$
$$+ q(1-e^{-b})\int_0^1 (e^{-ab/(1-a)}/(1-a))\,da$$
$$+\left[q(1-q)\sum_{\nu=1}^{\infty}(\nu!)^{-1}\prod_{j=2}^{\nu}(1-q/j)\int_0^b u^{\nu}e^{-u}\,du\right]\int_0^1 (e^{-ab/(1-a)}/(1-a))\,da,$$

which gives (3.4).

3.3. *Case* 3. $q(x, t) = q(x)(p(x))^t$, $t \geqq 0$, $0 < q(x)$, $p(x) < 1$. We state, without proof, the following result.

Theorem 3.2. For Case 3:

(i) *Form of the optimal rule.* If $p(x)$ is non-increasing in x, then there are numbers $\{d(j, n)\}_{j=1}^n$, $d(j, n) = d(j, n, p(\cdot), q(\cdot))$, decreasing in j, such that an optimal procedure is to solicit the largest among $X_1, \cdots, X_k$ if $d(k-1, n) \geqq L_{k-1}$, $L_k > d(k, n)$ and if unsuccessful to solicit each candidate that appears in turn until successful or until all observations are exhausted.

(ii) *Formula for decision numbers* $d(k, n)$. $d = d(k, n)$ satisfies

$$1 = \sum_{\nu=1}^{n-k} \binom{n-k}{\nu} (1-d)^\nu d^{-\nu} (1 + p(d)\nu/n(1-p(d)))g(n, \nu, d).$$

(iii) *Probability of successful solicitation.* The probability of choosing the largest observation using the optimal procedure is

$$P(n) = \sum_{k=1}^{n} \{P_1(k, n) + P_2(k, n) + P_3(k, n)\}$$

where

$$\begin{aligned} P_1(k, n) = &\int_{d(k,n)}^{d(k-1,n)} (x^{n-1}q(x)p(x)(1-(p(x))^{k-1})/(1-p(x)))\,dx \\ &+ \sum_{\nu=1}^{n-k} \binom{n-k}{\nu} \int_{d(k,n)}^{d(k-1,n)} (k-1-q(x)p(x)(1-(p(x))^{k-1})/(1-p(x))) \\ &\times x^{n-\nu-1}(1-x)^\nu g(n, \nu, x)\,dx, \qquad k \geqq 2 \\ = &\,0, \qquad k = 1, \end{aligned}$$

and $P_2(k, n)$, $P_3(k, n)$ are as in Theorem 3.1.

(iv) *Asymptotic formula for decision numbers.* If $\lim_{x\to 1} p(x) = p$, $\lim_{x\to 1} q(x) = q$, then $d(k, n) = 1 - b_{n-k}/(n-k)$ where $\lim_{j\to\infty} b_j = b$ with b satisfying

$$q^{-1} = \sum_{\nu=1}^{\infty} \prod_{j=2}^{\nu} (1 - q/j) b^\nu/\nu!$$

(v) *Asymptotic probability of successful solicitation.*

$$\begin{aligned} \lim_{n\to\infty} P(n) = &\left[q \sum_{\nu=1}^{\infty} \prod_{j=2}^{\nu} (1-q/j) b^\nu/\nu!\right]\left[e^{-b} - b\int_1^\infty (e^{-bv}/v)\,dv\right] \\ &+ q(e^b - 1)\int_1^\infty (e^{-bv}/v)\,dv \\ &+ \left[q(1-q) \sum_{\nu=1}^{\infty} (\nu!)^{-1} \prod_{j=2}^{\nu} (1-q/j) \int_0^b u^\nu e^{-u}\,du\right] e^b \int_1^\infty (e^{-bv}/v)\,dv. \end{aligned}$$

Remarks.

(i) As in Case 2 there are decision numbers $d(k, n)$ in this case as well. The difference in the form of the rule in Case 3 is that as soon as L_k exceeds $d(k, n)$ one makes a solicitation. This is because $q(x, t)$ is decreasing in t and one wishes to avoid decreasing the probability of a successful solicitation.

(ii) The assumption that $p(x)$ is non-increasing corresponds to a lower marginal probability of successful solicitation (when waiting one more observation) for larger observations. Thus, to use the employment/applicant interpretation, the better the applicant the relatively greater is the loss in his probability of acceptance by waiting one more interview before offering him the position.

(iii) If $q(x) \equiv q, p(x) \equiv p$ then, as in the remarks after Theorem 3.1 $g(n, \nu, x) = q \prod_{j=2}^{\nu} (1 - q/j)$.

(iv) The asymptotic results in (iv) and (v) are the same as for the case in which $q(x) \equiv q, p(x) \equiv 0$.

4. Some general considerations

The form of the optimal procedure for each of the functions $q(\cdot, \cdot)$ considered in Section 3 was obtained explicitly, in part because each procedure solicited a past observation at most once. In general this simplicity will be lacking and we must resort to Equations (2.1)–(2.5) to solve the problem.

We can, however, describe the general behavior of optimal procedures, at least to some extent. The results of this section are attempts at such a description.

The first three theorems are analogues in our full information setting of theorems given by Yang (1974) and Petruccelli (1981) in the case of no information.

The idea of Theorem 4.1 is that if one has the possibility of successfully soliciting an observation at a later date, one will be more demanding before soliciting a present observation than one would be if no future solicitation were possible.

Theorem 4.1. Let $\{d(k, n)\}_{k=1}^{n}$ be the decision numbers for the optimal procedure of Theorem 3.1 with $p(x) \equiv 0$, $q(x) \geqq 0$. Consider the optimal selection procedure for the problem defined by arbitrary $q(\cdot, \cdot)$ with $q(x, 0) = q(x)$. Then in state (x, k, t) with $x \leqq d(k, n)$, this procedure will always observe X_{k+1} rather than solicit L_k.

Proof. Let δ, δ_f, δ_b be as previously defined. Let ζ, ζ_f, ζ_b be the corresponding quantities when $q(x, t) = 0$, $t \geqq 1$, $q(x, 0) = q(x)$. We first show

$$\zeta(x, l, 0) < \delta(x, l, 0), x \in (0, 1), \qquad l = 1, \cdots, n-1. \tag{4.1}$$

(4.1) is easy to demonstrate for $l=n-1$. Suppose it is true for $l=n-1,\cdots,n-k+1$, for some $1\leqq k\leqq n$. Then

$$\begin{aligned}\delta_b(x,n-k,0)&=x^{n-k}q(x,0)+(1-q(x,0))\sum_{j=0}^{k-1}x^j\int_x^1\delta(y,n-k+j+1,0)\,dy\\&>x^{n-k}q(x,0)+(1-q(x,0))\sum_{j=0}^{k-1}x^j\int_x^1\zeta(y,n-k+j+1,0)\,dy\\&=\zeta_b(x,n-k,0).\end{aligned}$$

This proves (4.1). Let $d=d(k,n)$ and assume $x\leqq d$. Then $\zeta_f(x,k,0)>\zeta_b(x,k,0)$ which implies

$$x^{n-k}<\sum_{j=0}^{n-k-1}x^j\int_x^1\zeta(y,n-k+j+1,0)\,dy$$

and so

$$\begin{aligned}\delta_f(x,k,t)&\geqq\sum_{j=0}^{n-k-1}x^j\int_x^1\delta(y,n-k+j+1,0)\,dy\\&>q(x,t)\sum_{j=0}^{n-k-1}x^j\int_x^1\zeta(y,n-k+j+1,0)\,dy\\&\quad+(1-q(x,t))\sum_{j=0}^{n-k-1}x^j\int_x^1\delta(y,n-k+j+1,0)\,dy\\&>q(x,t)x^{n-k}+(1-q(x,t))\sum_{j=0}^{n-k-1}x^j\int_x^1\delta(y,n-k+j+1,0)\,dy\\&=\delta_b(x,k,t)\end{aligned}$$

Theorem 4.2 gives necessary and sufficient conditions, involving L_{n-1}, for the optimal strategy to view all observations before making a solicitation.

Theorem 4.2. In order that, under the optimal strategy, an offer is never made until all applicants have been interviewed, it is:

(i) Necessary that

$$\int_{L_{n-1}}^1 q(y,0)\,dy\geqq L_{n-1}(1-q(L_{n-1},t+1)/q(L_{n-1},t)),\qquad 0\leqq t\leqq n-1. \tag{4.2}$$

(ii) Sufficient that

$$\int_x^1 q(y,0)\,dy\geqq x(1-q(x,t+1)/q(x,t)),\qquad 0\leqq t\leqq n-1,\ x\leqq L_{n-1}. \tag{4.3}$$

Proof. It is easy to show that

$$\delta_f(L_{n-1},n-1,t)\geqq\delta_b(L_{n-1},n-1,t),\qquad 0\leqq t\leqq n-1$$

iff (4.2) holds. This proves (i).

To prove (ii) assume (4.3) is true and that

$$\delta_f(x, j, t) \geqq \delta_b(x, j, t) \qquad 0 \leqq t < j, \qquad j = n-1, \cdots, n-k+1.$$

Then

$$\begin{aligned}\delta_f(x, n-k, t) &= \sum_{j=0}^{k-1} x^j \int_x^1 \delta(y, n-k+j+1, 0)\, dy + x^k q(x, t+k)\\ &\geqq q(x, t)x^k + (1-q(x, t)) \sum_{j=0}^{k-1} x^j \int_x^1 \delta(y, n-k+j+1, 0)\, dy\\ &= \delta_b(x, n-k, t)\end{aligned}$$

iff

$$\sum_{j=0}^{k-1} x^j \int_x^1 \delta(y, n-k+j+1, 0)\, dy \geqq x^k(1-q(x, t+k)/q(x, t)).$$

By assumption

$$\sum_{j=1}^{k-1} x^j \int_x^1 \delta(y, n-k+j+1, 0)\, dy \geqq x^k(1-q(x, t+k-1)/q(x, t)).$$

Thus it suffices to show

$$x^{-k} \int_x^1 \delta(y, n-k+1, 0)\, dy \geqq (q(x, t+k-1) - q(x, t+k))/q(x, t). \tag{4.4}$$

Since $\delta(y, n-k+1, 0) \geqq q(y, 0)y^{k-1}$, the left side of (4.4) exceeds

$$\begin{aligned}x^{-k} \int_x^1 q(y, 0)y^{k-1}\, dy \geqq x^{-1} \int_x^1 q(y, 0)\, dy&\\ \geqq (1-q(x, t+k)/q(x, t+k-1))&\\ \geqq (q(x, t+k-1)/q(x, t))(1-q(x, t+k)/q(x, t+k-1)).&\end{aligned}$$

Theorem 4.3. Let $h(n, x) \equiv 0$, $x \in [0, 1]$. For $x \in (0, 1]$ and $s = 1, \cdots, n-1$ let

$$h(s, x) = \sum_{\nu=1}^{n-s} \binom{n-s}{\nu} (1-x)^\nu x^{-\nu} g(n, \nu, x)$$

where $g(n, \nu, x)$ is as given in Section 3 with $q(x) = q(x, 0)$. Assume g is non-increasing in x. Let the decision numbers $\{d(k, n)\}_{k=1}^n$ be as in Theorem 4.1. Then:

(i) If for some $1 \leqq s < n$, $L_s = x > d(s, n)$ and

$$q(y, t+1)/q(y, t) \leqq (1-h(s, y))/(1-h(s+1, y)), \qquad 0 \leqq t < n,\ y \geqq x, \tag{4.5}$$

the optimal procedure will solicit the current largest observation (if previously unsolicited) at each time $s, \cdots, n$.

(ii) Further, if $q(y, t+1)/q(y, t)$ is non-increasing in y for each $t<n$ and if (4.5) holds for $y=x$ and all $t<n$ then the conclusion of (i) holds.

Proof. It can be shown that the right side of (4.5) is increasing in s and y for $y \geqq d(s, n)$ (see Appendix B). This means (ii) follows from (i), so we consider the latter.

Clearly (i) is true for $s=n-1$ since

$$g(n, 1, x)=(1-x)^{-1}\int_x^1 q(y, 0)\,dy.$$

Assume the result true for $s=n-1, n-2, \cdots, k+1$, that $L_k=x>d(k, n)$ and that (4.5) holds for $s=k$. Under these assumptions

$$h(k, x)=x^{k-n}\delta(x, k, \infty), \qquad h(k+1, x)=x^{k+1-n}\delta(x, k+1, \infty).$$

Thus

$$\begin{aligned}\delta_f(x, k, t)&=\int_x^1 \delta(y, k+1, 0)\,dy+x\delta(x, k+1, t+1)\\&=\int_x^1 \delta(y, k+1, 0)\,dy+x^{n-k}q(x, t+1)\\&\quad+x(1-q(x, t+1))\delta(x, k+1, \infty)\\&=x^{n-k}[q(x, t+1)+h(k, x)-q(x, t+1)h(k+1, x)].\\\delta_b(x, k, t)&=x^{n-k}q(x, t)+(1-q(x, t))\delta(x, k, \infty)\\&=x^{n-k}[q(x, t)+(1-q(x, t))h(k, x)].\end{aligned}$$

So

$$\delta_f(x, k, t)-\delta_b(x, k, t)=x^{n-k}[q(x, t+1)(1-h(k+1, x))-q(x, t)(1-h(k, x))].$$

Remark. If $q(x, t)=q(t)$ the condition that g be non-increasing in x is automatically satisfied.

For the next theorem we assume, contrary to what is assumed in the rest of the paper, that $q(x, t)$ may be non-decreasing in t.

Theorem 4.4. Assume $q(y, 0)\neq 0$. Then $\delta_f(x, k, t)>\delta_b(x, k, t)$ $x\in[0, 1]$, $k>t\geqq t_0$, $t_0\leqq n-1$ if and only if

$$q(x, t+1)/q(x, t)\geqq 1 \qquad x\in[0, 1],\ t_0\leqq t\leqq n-2.$$

Proof. Sufficiency is easy to show using (2.1)–(2.5). To show necessity let $q=\sup_{y\in[0,1]} q(y, 0)$.

$$\delta_f(x, n-1, t)>\delta_b(x, n-1, t) \quad \text{iff} \quad \int_x^1 q(y, 0)\,dy>x(1-q(x, t+1)/q(x, t))$$

which implies

$$q(1-x)>x(1-q(x,t+1)/q(x,t)).$$

If $q(x,t+1)/q(x,t)<1$ we have from above that $\delta_f(x,n-1,t)>\delta_b(x,n-1,t)$ only if $x\leqq q/(1+q-q(x,t+1)/q(x,t))<1$ which contradicts the assumption that $x\in[0,1]$.

Remark. Theorem 4.4 shows that if we require $q(x,t)$ to be non-increasing in t then all observations are always taken before any solicitation is made if and only if $q(x,t)\equiv q(x)$, $t\geqq 0$.

If the marginal (in time) probability of a successful solicitation is negative and decreasing with the size of the observation it would seem reasonable that for any state (x,k,t), $t<\infty$, an optimal procedure would solicit a current candidate, as opposed to observing the next observation, if x is large. This is precisely the case as shown by the following result.

Theorem 4.5. Suppose $1>q(x,t+1)/q(x,t)>0$ is non-increasing in x for each $0\leqq t<n$. Then if $q(x,t)$ is continuous in x there are numbers $\{d(n,j,t)\}_{0\leqq t<j\leqq n}$ such that in state (x,k,t) the optimal procedure will solicit x_{k-t} if and only if $x>d(n,k,t)$.

Proof. Let $A(n,j,t)=\{x:\delta_f(x,j,t)>\delta_b(x,j,t)\}$. Suppose

$$x\in\bigcap_{j=1}^{k-1}A(n,n-k+j,t).$$

Then

$$\begin{aligned}(\delta_f(x,n-k,t)-\delta_b(x,n-k,t))/x^kq(x,t)(1-q(x,t+n-k)/q(x,t))\\ =(1-q(x,t+n-k)/q(x,t))^{-1}x^{-k}\delta(x,n-k,\infty)-1\end{aligned}$$

which is non-increasing in x.

For $x\notin\bigcap_{j=1}^{k-1}A(n,n-k+j,t)$, let $s=s(x)$ be the smallest integer $0\leqq s\leqq k-2$ for which $x\notin A(n,n-k+s+1,t)$. Then

$$\begin{aligned}&(\delta_f(x,n-k,t)-\delta_b(x,n-k,t))/x^kq(x,t)(1-q(x,t+s+1)/q(x,t))\\ &=(1-q(x,t+s+1)/q(x,t))^{-1}\sum_{j=0}^{s}x^{j-k}\int_x^1\delta(y,n-k+j+1,0)\,dy\\ &\quad+\sum_{j=0}^{k-s-2}x^{j+s+1-k}\int_x^1\delta(y,n-k+s+j+2,0)\,dy-1\end{aligned}$$

which is non-increasing in x.

The result follows by the continuity of δ_b and δ_f in x.

The final two theorems characterize the probabilities of successful solicitation in the cases for which, in some sense, the optimal procedure solicits independently of t at each stage k.

Theorem 4.6. If $q(x, t) = q(t)$, independent of x, then at (x, k, t), $0 \leqq t < k \leqq n-1$, the optimal procedure solicits independently of t if and only if $q(t) = qp^t$ for some $0 < q$, $p \leqq 1$.

Proof. It is easy to show, using (2.1)–(2.5), that $\delta_f(x, k, t) - \delta_b(x, k, t)$ is greater than 0 independently of t for each k. This gives sufficiency. To show necessity we note that $\delta_b(x, n-1, t) > \delta_f(x, n-1, t)$ iff $q(0)x^{-1}(1-x) < 1 - q(t+1)/q(t)$ which is independent of t iff $q(t+1)/q(t)$ is independent of t, $0 \leqq t < n-1$. Let $q = q(0)$ and $p = q(1)/q(0)$. The result follows.

Theorem 4.7. $(\delta_b(x, n-1, t) - \delta_f(x, n-1, t))/q(x, t)$ is independent of t if and only if $q(x, t) = q(x)(p(x))^t$.

Proof. Similar to the above.

Appendix A

This appendix derives a computational formula for

$$g(n, \nu, x) = E(q(Z_1) \prod_{j=2}^{c(\nu)} (1 - q(Z_j)) \mid \nu, X_k = L_k = x).$$

Let $Y_1, \cdots, Y_\nu$ denote the ν exceedances of x among $X_{k+1}, \cdots, X_n$, in order of observation. Let $U_j = \max\{Y_1, \cdots, Y_j\}$ for each $1 \leqq j \leqq \nu$. Let $\xi_1 = \xi_1(\nu) = kI(Y_k = U_\nu)$, $\xi_{c(\nu)} = \xi_{c(\nu)}(\nu) = 1$. $\xi_j = \xi_j(\nu) = kI(Y_k = U_{\xi_{j-1}-1})$, $j = 2, \cdots, c(\nu) - 1$ so that ξ_1 is the time at which the largest Y occurs, ξ_2 the time at which the previous largest occurs, etc.

In what follows the theory of record values of a random sequence is used. See Haghighi-Talab and Wright (1973) for some details of this theory.

Let $|S_k^j|$ be the modulus of the indicated Stirling number of the first kind. Then

$$\begin{aligned} P(c(\nu) = j \mid \xi_1, \nu) &= |S_{\xi_1-1}^{j-1}|/(\xi_1 - 1)! \qquad && 2 \leqq j \leqq \xi_1 \\ &= 0 && \xi_1 < j. \end{aligned}$$

Note the first quantity is the probability of $j-1$ record values among observations $Y_1, \cdots, Y_{\xi_1-1}$. Note also that Y_1 is always a record value so that given $\xi_1 > 1$, $c(\nu) \geqq 2$. Now assume $c(\nu) \geqq 2$

$$P(\xi_1 = l \mid \nu) = \nu^{-1} \qquad 1 \leqq l \leqq \nu$$

and

$$P(c(\nu) = j \mid \nu) = |S_\nu^j|/\nu!$$

so that

$$P(\xi_1 = l \mid c(\nu), \nu) = (\nu-1)!\,|S_{l-1}^{c(\nu)-1}|/(l-1)!\,|S_\nu^{c(\nu)}| \qquad c(\nu) \leqq l \leqq \nu.$$

To obtain $P(\xi_k = l \mid \xi_1, \cdots, \xi_{k-1}, c(\nu), \nu)$ we note that this is the probability that the largest among $Y_1, \cdots, Y_{\xi_{k-1}-1}$ occurs at l given there are $c(\nu)-k+1$ records among these observations. Thus for $k = 2, \cdots, c(\nu)-1$

$$\begin{aligned} &P(\xi_k = l \mid \xi_1, \cdots, \xi_{k-1}, c(\nu), \nu) \\ &\quad = (\xi_{k-1}-2)!\,|S_{l-1}^{c(\nu)-k}|/(l-1)!\,|S_{\xi_{k-1}-1}^{c(\nu)-k+1}| \qquad c(\nu)-k+1 \leqq l \leqq \xi_{k-1}-1. \end{aligned}$$

So we obtain a probability mass function

$$\begin{aligned} &f(\xi_1, \cdots, \xi_{c(\nu)-1} \mid c(\nu), \nu, X_k = L_k = x) = f(\xi_1, \cdots, \xi_{c(\nu)-1} \mid c(\nu), \nu) \\ &\quad = (\nu-1)!\,|S_{\xi_{c(\nu)-1}-1}^{1}|/|S_\nu^{c(\nu)}|\,(\xi_{c(\nu)-1}-1)! \prod_{i=1}^{c(\nu)-2} (\xi_i - 1), \\ &\quad = (\nu-1)! \Big/ \prod_{i=1}^{c(\nu)-2} (\xi_i - 1), \qquad c(\nu) \leqq \xi_1 \leqq \nu,\ c(\nu)-j+1 \leqq \xi_j \leqq \xi_{j-1}-1, \\ &\qquad\qquad j = 2, \cdots, c(\nu)-1. \end{aligned}$$

Now given $\xi_1, \cdots, \xi_{c(\nu)-1}$, $c(\nu)$, ν, $X_k = L_k = x$, $Z_1, \cdots, Z_{c(\nu)}$ have density

$$\begin{aligned} &h(z_1, \cdots, z_{c(\nu)} \mid \xi_1, \cdots, \xi_{c(\nu)-1}, c(\nu), \nu, X_k = L_k = x) \\ &\quad = \left[\nu(z_1 - x)^\nu/(1-x)^\nu (z_{c(\nu)} - x)^2 \prod_{i=2}^{c(\nu)-1} (z_i - x)\right] \\ &\qquad \times \prod_{j=1}^{c(\nu)-1} (\xi_j - 1)((z_{j+1} - x)/(z_j - x))^{\xi_j}, \qquad x < z_{c(\nu)} < \cdots < z_1 < 1. \end{aligned}$$

This follows from the fact that Z_1 is distributed as the largest observation from ν $U[x, 1]$ random variables and, given $c(\nu)$, $X_k = L_k = x$, $\xi_1, \cdots, \xi_{c(\nu)-1}$, and $Z_1, \cdots, Z_{j-1}$, Z_j is distributed as the largest from among $\xi_{j-1}-1$ $U[x, Z_{j-1}]$ random variables, $j = 2, \cdots, c(\nu)$.

Thus we have the density

$$\begin{aligned} &f(z_1, \cdots, z_{c(\nu)} \mid c(\nu), \nu, X_k = L_k = x) \\ &\quad = \left[\nu!(z_1 - x)^\nu/(1-x)^\nu\,|S_\nu^{c(\nu)}|\,(z_{c(\nu)} - x)^2 \prod_{i=2}^{c(\nu)-1} (z_i - x)\right] \\ &\qquad \times \sum_{\xi_1 = c(\nu)}^{\nu} \left(\frac{z_2 - x}{z_1 - x}\right)^{\xi_1} \sum_{\xi_2 = c(\nu)-1}^{\xi_1 - 1} \left(\frac{z_3 - x}{z_2 - x}\right)^{\xi_2} \cdots \\ &\qquad \times \sum_{\xi_{c(\nu)-2} = 3}^{\xi_{c(\nu)-3}-1} \left(\frac{z_{c(\nu)-1} - x}{z_{c(\nu)-2} - x}\right)^{\xi_{c(\nu)-2}} \\ &\qquad \times \sum_{\xi_{c(\nu)-1} = 2}^{\xi_{c(\nu)-2}-1} \left(\frac{z_{c(\nu)} - x}{z_{c(\nu)-1} - x}\right)^{\xi_{c(\nu)-1}}. \end{aligned}$$

If $c(\nu)=1$

$$f(z_1 \mid c(\nu)=1, \nu, X_k=L_k=x)=\nu(z_1-x)^{\nu-1}/(1-x)^{\nu}, \qquad x<z<1.$$

Now since

$$P(c(\nu)=l \mid \nu, X_k=L_k=x)=P(c(\nu)=l \mid \nu)=|S_\nu^l|/\nu!$$

we have

$$g(n,\nu,x)=\sum_{l=1}^{\nu}(\nu!)^{-1}|S_\nu^l|\int_x^1 q(z_1)\int_x^{z_1}(1-q(z_2))\cdot\cdots$$
$$\times\int_x^{z_{l-1}}(1-q(z_l))f(z_1,\cdots,z_l \mid \nu, c(\nu)=l, X_k=L_k=x)\,dz_l\cdot\cdots\cdot dz_1.$$

Appendix B

This appendix proves the following result which is used in Theorem 4.3: $G(s,y)=(1-h(s,y))/(1-h(s+1,y))$ is increasing in s and y for $y\geqq d(s,n)$. We shall need the following lemma, the proof of which is easy.

Lemma. Let $g>f>0$ be differentiable functions on $(0,\infty)$ with $f'>g'>0$. Then $(f/g)'>0$ on $(0,\infty)$.

Now

$$h(s+1,y)-h(s,y)=\sum_{\nu=1}^{n-s-1}\binom{n-s-1}{\nu}(1-y)^{\nu}y^{-\nu}g(n,\nu,y)$$
$$-\sum_{\nu=1}^{n-s}\binom{n-s}{\nu}(1-y)^{\nu}y^{-\nu}g(n,\nu,y)<0.$$

Thus $h(s,y)$ is decreasing in s.

$$G(s,y)>G(s-1,y)\quad\text{iff}\quad (h(s,y)-h(s+1,y))/(1-h(s+1,y))$$
$$<(h(s-1,y)-h(s,y))/(1-h(s,y)).$$

This holds in particular if $h(s,y)-h(s+1,y)<h(s-1,y)-h(s,y)$ or equivalently

$$\sum_{\nu=1}^{n-s}\binom{n-s-1}{\nu-1}(1-y)^{\nu}y^{-\nu}g(n,\nu,x)<\sum_{\nu=1}^{n-s+1}\binom{n-s}{\nu-1}(1-y)^{\nu}y^{-\nu}g(n,\nu,x)$$

which is clearly true. Thus G is increasing in s.

Let $z=(1-y)/y$ and let

$$v(s,z)=\sum_{\nu=1}^{n-s}\binom{n-s}{\nu}z^{\nu}g(n,\nu,(z+1)^{-1})$$
$$\frac{\partial v}{\partial z}=\sum_{\nu=1}^{n-s}\binom{n-s}{\nu}z^{\nu-1}\left[z\frac{\partial}{\partial z}g(n,\nu,(z+1)^{-1})+\nu g(n,\nu,(z+1)^{-1})\right]$$
$$>0\quad\text{since}\quad\frac{\partial}{\partial x}g(n,\nu,x)\leqq 0.$$

So v is increasing in z and furthermore

$$\frac{\partial}{\partial z} v(s, z) > \frac{\partial}{\partial z} v(s+1, z).$$

But this means that $f(y) = 1 - h(s, y)$, $g(y) = 1 - h(s+1, y)$ satisfy the conditions of the lemma and so $G(s, y)$, their quotient, is increasing in y.

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