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ON THE BEST-CHOICE PROBLEM WHEN THE NUMBER OF OBSERVATIONS IS RANDOM

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Abstract

We consider the problem of maximizing the probability of choosing the largest from a sequence of N observations when N is a bounded random variable. The present paper gives a necessary and sufficient condition, based on the distribution of N, for the optimal stopping rule to have a particularly simple form: what Rasmussen and Robbins (1975) call an $s(r)$ rule. A second result indicates that optimal stopping rules for this problem can, with one restriction, take virtually any form.

OPTIMAL STOPPING; SECRETARY PROBLEM; RELATIVE RANKS; OPTIMIZATION THEORY

1. Introduction

We consider the following problem: N observations are taken sequentially with the object of choosing the largest. After each observation a decision must be made, based only on the relative ranks of the observations seen so far, to choose or reject that observation. Once rejected an observation is no longer available. If N is known this is a version — often called the best-choice problem — of the well-known secretary problem (see Gilbert and Mosteller (1966)).

In this paper we allow N to be a bounded random variable. Section 2 seeks to describe the structure of optimal rules and in particular when an optimal rule will only accept the first observation of relative rank 1 appearing after a given time. This is what Rasmussen and Robbins (1975) call an $s(r)$ rule.

A more general theorem of Presman and Sonin (1972) provides, in our setting, a simple sufficient condition for an $s(r)$ rule to be optimal. This same sufficient condition can also be deduced from a proposition of Irle (1980). Theorem 2.2 refines this condition. The necessary and sufficient condition of Theorem 2.3 follows from equations of Presman and Sonin.

In Section 3 we prove that optimal stopping rules for this problem can take, with one restriction, literally any form.

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Rasmussen and Robbins (1975) considered the above problem when N is a bounded random variable and Rasmussen (1975) considered the problem for an arbitrary (as opposed to 0–1) payoff based on the rank of the observation chosen, with N as above. However, as will be explained in Section 2, both papers state erroneous results concerning the structure of optimal stopping rules.

The result of Section 3 is the most general possible counterexample to these erroneous results.

Similar problems have been considered by Gianini-Pettitt (1979).

2. When is an $s(r)$ rule optimal?

Assume $N \leqq n$, $P(N = k) = p_k$, $k = 1, \cdots, n$ with $p_n > 0$. Following Rasmussen and Robbins we define an $s(r)$ *rule* to be a stopping rule of the form: for some $2 \leqq r \leqq n$ observe the first $r - 1$ observations and then choose the first among the remaining observations which is the best observed thus far. It is well known that if $P(N = n) = p_n = 1$, the optimal rule is an $s(r)$ rule with

$$r = t(n) = \min \left\{ l \geqq 1 : \sum_{m=l}^{n-1} 1/m \leqq 1 \right\}.$$

Rasmussen and Robbins incorrectly assume Theorem 3.1 of Rasmussen to be true. (In fact the error occurs in Rasmussen's Lemma 3.2 to Theorem 3.1.) Hence their statement that the optimal rule is an $s(r)$ rule no matter what the distribution of N is incorrect.

Irle (1980) shows by counterexample that whether an optimal rule is an $s(r)$ rule depends on the distribution of N. Using a more general formulation of the problem he derives several sufficient conditions on the distribution of N for the optimal rule to be an $s(r)$ rule. One of these becomes, under our assumptions,

$$p_k > \sum_{m=k+1}^{n} p_m/m \text{ implies } p_l > \sum_{m=l+1}^{n} p_m/m, \qquad l > k. \tag{2.1}$$

This condition is also a special case of Theorem 3.2 of Presman and Sonin (1972).

This condition is weakened somewhat in Theorem 2.2 below.

The most general possible counterexample to the incorrect Rasmussen result in the best-choice case (equivalently, the incorrect Rasmussen and Robbins result) is derived in Section 3 below.

2.1. *Mathematical formulation.* For $k = 1, \cdots, n$ observation k will be called a *candidate* if it is the largest of observations $1, \cdots, k$. It is assumed that at the time of observation we are only told the rank of each observation relative to those previously observed.

Let $a(k)$ and $b(k)$ denote the probability, when observation k is a candidate, of selecting the largest of all N observations by observing the $(k+1)$th observation and behaving optimally thereafter and by selecting observation k, respectively. Let $m(k) = \max\{a(k), b(k)\}$.

Let

$$\alpha(k) = \left(\sum_{j=k}^{n} p_j/k\right) a(k)$$

$$\beta(k) = \left(\sum_{j=k}^{n} p_j/k\right) b(k)$$

$$\mu(k) = \max\{\alpha(k), \beta(k)\} = \left(\sum_{j=k}^{n} p_j/k\right) m(k).$$

Suppose observation k has just been seen and is a candidate. Let

$A = \{$largest of all N is chosen by observing observation $k+1$ and continuing optimally$\}$

$B = \{$observation $k+1$ is a candidate$\}$

$$\begin{aligned} a(k) &= P(A \mid N \geqq k) = P(AB\{N \geqq k+1\} \mid N \geqq k) + P(AB^c\{N \geqq k+1\} \mid N \geqq k) \\ &= P(A \mid B, N \geqq k+1)P(B \mid N \geqq k+1)P(N \geqq k+1 \mid N \geqq k) \\ &\quad + P(A \mid B^c, N \geqq k+1)P(B^c \mid N \geqq k+1)P(N \geqq k+1 \mid N \geqq k) \\ &= (m(k+1))(1/(k+1))\left(\sum_{j=k+1}^{n} p_j \bigg/ \sum_{j=k}^{n} p_j\right) \\ &\quad + a(k+1)(k/(k+1))\left(\sum_{j=k+1}^{n} p_j \bigg/ \sum_{j=k}^{n} p_j\right). \end{aligned}$$

This implies:

$$\alpha(k) = \alpha(k+1) + \mu(k+1)/k$$

$$\begin{aligned} b(k) &= P(\text{observation } k \text{ is largest of all } N \mid N \geqq k) \\ &= \sum_{j=k}^{n} P(\text{observation } k \text{ largest of all } j \mid N = j,\ N \geqq k)P(N = j \mid N \geqq k) \\ &= \sum_{j=k}^{n} (k/j)\left(p_j \bigg/ \sum_{l=k}^{n} p_l\right). \end{aligned}$$

Thus we obtain the following relations:

$$\alpha(k) = \sum_{j=k+1}^{n} \mu(j)/(j-1), \qquad 1 \leqq k \leqq n-1 \tag{2.2}$$

$$\beta(k) = \sum_{j=k}^{n} p_j/j, \qquad 1 \leqq k \leqq n \tag{2.3}$$

(2.4) $$\mu(n) = \beta(n) = p_n/n.$$

It is easily shown that $\beta(n-1) > \alpha(n-1)$ and that if

$$\beta(j) > \alpha(j), \qquad l+1 \leqq j \leqq n-1,$$

then

$$\beta(l) - \alpha(l) = p_l/l + \sum_{j=l+1}^{n} (p_j/j)\left(1 - \sum_{m=l}^{j-1} 1/m\right).$$

From this we obtain the following result.

Theorem 2.1. Let $t(n) = \min\{l \geqq 1 : \Sigma_{m=l}^{n} 1/m \leqq 1\}$. In the best-choice problem with random $N \leqq n$, the first time the optimal rule will select a candidate occurs no later than the $t(n)$th observation. Further, the optimal rule always selects the first candidate among observations $t(n), \cdots, n$ if stopping has not previously occurred.

2.2. *Necessary and sufficient conditions for the optimal rule to be an* $s(r)$ *rule.* Using (2.2)–(2.4) we obtain

(2.5) $$\beta(k) - \alpha(k) = \beta(k+1) - \alpha(k+1) + p_k/k - \mu(k+1)/k,$$

so that

$$\beta(k+1) > \alpha(k+1) \quad \text{and} \quad p_k \Big/ \sum_{m=k+1}^{n} p_m/m > 1$$

imply $\beta(k) > \alpha(k)$. But we know from Theorem 2.1 that $\beta(l) > \alpha(l)$, $l \geqq t(n)$. Thus if $k \leqq t(n)-1$ we need only require

$$p_l \Big/ \sum_{m=l+1}^{n} p_m/m > 1, \qquad k \leqq l \leqq t(n)-1$$

in order that $\beta(l) > \alpha(l)$, $l \geqq k$.

If

$$\xi = \max\left\{k \geqq 1 : p_k \Big/ \sum_{m=k+1}^{n} p_m/m \leqq 1\right\}$$

and if we adopt condition (2.6) below then

$$p_l \Big/ \sum_{m=l+1}^{n} p_m/m \leqq 1, \qquad 1 \leqq l \leqq \xi,$$

and so from (2.5) we have

$$\begin{aligned} \beta(l) - \alpha(l) &\leqq \beta(l+1) - \alpha(l+1), \qquad 1 \leqq l \leqq \min\{\xi, t(n)-1\} \\ &> \beta(l+1) - \alpha(l+1), \qquad \min\{\xi, t(n)-1\} \leqq l \leqq t(n)-1. \end{aligned}$$

Thus condition (2.6) guarantees that $\alpha(l) > \beta(l)$ implies $\alpha(j) > \beta(j)$, $1 \leqq j \leqq l$, and hence that the optimal rule is an $s(r)$ rule. Thus we have the following result.

Theorem 2.2. Let $t(n) = \min\{l \geqq 1 : \Sigma_{m=l}^{n-1} 1/m \leqq 1\}$. Then a sufficient condition for the optimal rule to be an $s(r)$ rule is

$$p_k \Big/ \sum_{m=k+1}^{n} p_m/m > 1 \text{ implies } p_l \Big/ \sum_{m=l+1}^{n} p_m/m > 1, \quad k \leqq l \leqq t(n) - 1. \tag{2.6}$$

If we assume in Presman and Sonin's (1972) formulation that $p_k = 0$, $k \geqq n+1$, then their

$$c_l = \sum_{m=l}^{n} p_m/m - \sum_{m=l+1}^{n} c(m, l)p_m,$$

where $c(m, l)$ is defined in Theorem 2.3 below, so that their Theorem 3.1 implies $\beta(k) > \alpha(k)$, $k \geqq k(n)+1$ and $\beta(k(n)) < \alpha(k(n))$, where $k(n)$ is defined in Theorem 2.3 below. Using their formulas (3.4) and (3.5) and an induction argument we have that

$$\alpha(l) > \beta(l), \qquad 1 \leqq l < k(n)$$

iff (2.7) below holds. Hence we have the following result.

Theorem 2.3. If $c(m, l) = (\Sigma_{j=l+1}^{m} 1/(j-1))/m$ and if

$$k(n) = \max\left\{l \geqq 1 : \sum_{m=l}^{n} p_m/m < \sum_{m=l+1}^{n} c(m, l)p_m\right\}$$

then the optimal rule is an $s(r)$ rule if and only if

$$l \sum_{m=l}^{n} p_m/m < k(n) \sum_{m=k(n)+1}^{n} c(m, k(n))p_m, \qquad 1 \leqq l \leqq k(n). \tag{2.7}$$

Further, if condition (2.7) holds the optimal rule passes over observations $1, \cdots, k(n)$ and chooses the first candidate thereafter.

Notice that (2.7) can be written as

$$l \sum_{m=l}^{k(n)-1} p_m/m < \sum_{m=k(n)+1}^{n} (k(n)c(m, k(n)) - l/m)p_m - lp_{k(n)}/k(n), \qquad 1 \leqq l \leqq k(n). \tag{2.8}$$

Given $k(n)$ the right side of (2.8) depends on $p_{k(n)}, \cdots, p_n$ while the left depends on $p_l, \cdots, p_{k(n)-1}$. $k(n)$ itself is determined by the values of p_j for larger j. Thus we can view condition (2.8) as: determine $k(n)$ using $p_n, p_{n-1}, \cdots, p_{k(n)}$. This determines a benchmark for each $l < k(n)$, namely the right side of (2.8). The optimal rule is an $s(r)$ if the p_j for $l \leqq j \leqq k(n) - 1$ are 'small enough' relative to the lth benchmark.

3. What form can an optimal rule take?

In this section we state and prove the most general counterexample to the false theorem of Rasmussen in the best-choice case. Specifically, we prove that, taking into account the restriction imposed by Theorem 2.1, optimal stopping rules for the best-choice problem with random N can take any form whatsoever.

Formally, let $\mathcal{S}_\tau \subset \{1, \cdots, t(n)-1\}$ be the set of positive integers less than $t(n)$ at which a stopping rule τ will stop — that is, integers j at which τ will select observation j if j is a candidate. Theorem 2.1 guarantees that an optimal rule will always stop on $\{t(n), \cdots, n\}$.

Theorem 3.1. Let A be a subset of $\{1, \cdots, t(n)-1\}$. There is a probability distribution for $N, p_1, \cdots, p_n$ with $p_n > 0$ such that if τ is an optimal stopping rule for the best-choice problem with N observations, then $\mathcal{S}_\tau = A$.

Proof. $\beta(k) \geqq \alpha(k)$, $t(n) \leqq k \leqq n$ by Theorem 2.1. For the purposes of this proof and without loss of generality it will be assumed that if $\beta(k) = \alpha(k)$ and if observation k is a candidate then observation k will be selected.

From (2.2)–(2.4) we see that for fixed $p_k, \cdots, p_n$, $\alpha(k)$, $\beta(k)$ and hence $\mu(k)$ can be written as linear combinations of $p_k, \cdots, p_n$. If for some $c > 0$ we replace the p_j by $p'_j = cp_j$, $k \leqq j \leqq n$, the resulting linear combinations expressing $\alpha(k)$, $\beta(k)$ and $\mu(k)$ change only by replacing p_j by p'_j. Thus we obtain

$$\alpha'(k) = c\alpha(k)$$

$$\beta'(k) = c\beta(k)$$

$$\mu'(k) = c\mu(k)$$

where α, β, μ are defined in Section 2 when the distribution of N is $\{p_j\}_{j=1}^n$ and α', β', μ' result by changing p_j to p'_j, $j \geqq k$. In particular the sign of $\beta(k) - \alpha(k)$ remains unchanged under rescaling. Now

$$\beta(k) - \alpha(k) = \sum_{j=k}^{n} p_j/j - \sum_{j=k+1}^{n} \mu(j)/(j-1)$$

$$= \beta(k+1) - \alpha(k+1) + p_k/k - \mu(k+1)/k.$$

Thus:

(i) If $\alpha(k+1) > \beta(k+1)$ we can, by appropriate rescaling of $p_{k+1}, \cdots, p_n$ and appropriate choice of p_k, make $\beta(k) - \alpha(k)$ positive, negative or 0 while retaining the form of the optimal rule at observations $k+1, \cdots, n$.

(ii) If $\alpha(k+1) = \beta(k+1) = \mu(k+1)$, then

$$\beta(k) - \alpha(k) = p_k/k - \beta(k+1)/k = \left(p_k - \sum_{j=k+1}^{n} p_j/j\right) \Big/ k$$

which can be made positive, negative, or 0 by appropriate rescaling of $p_{k+1},\cdots,p_n$ for any choice of $p_k>0$. As in (i) such rescaling does not affect the form of the optimal rule for observations $k+1,\cdots,n$.

For any subset $A\subset\{1,\cdots,t(n)-1\}$ we can choose $p_1,\cdots,p_n$ so that for an optimal stopping rule τ, $\mathcal{S}_\tau=A$, by using the following procedure:

1. If $A\neq\emptyset$ there is an $1\leqq r\leqq t(n)-1$ such that $r=\max\{k:k\in A\}$. Then we want

$$\begin{aligned}&\alpha(k)>\beta(k), \qquad r<k\leqq t(n)-1\\&\alpha(r)=\beta(r).\end{aligned}\tag{3.1}$$

By successive use of (i) above we may obtain (3.1). Then we may use (ii) to make $\alpha(r-1)>\beta(r-1)$ if $r-1\notin A$ or to make $\alpha(r-1)=\beta(r-1)$ if $r-1\in A$. We may continue in this manner until we obtain

$$\begin{aligned}&\alpha(k)=\beta(k), \qquad k\in A\\&\alpha(k)>\beta(k), \qquad k\notin A\end{aligned}$$

which guarantees $\mathcal{S}_\tau=A$.

Note that a final rescaling may be necessary to ensure that $\Sigma_{j=1}^{n}p_j=1$.

2. If $A=\emptyset$ choose $p_n=1$, $p_j=0$, $j<n$.

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