The Annals of Statistics
1980, Vol. 8, No. 5, 1171–1174

ON A BEST CHOICE PROBLEM WITH PARTIAL INFORMATION

BY JOSEPH D. PETRUCCELLI

Worcester Polytechnic Institute

For a given family $\mathcal{F}$ of continuous cdf's n i.i.d. random variables with cdf $F \in \mathcal{F}$ are observed sequentially with the object of choosing the largest. An upper bound for the greatest asymptotic probability of choosing the largest is $\alpha \doteq .58$, the optimal asymptotic value when F is known, and a lower bound is e^{-1}, the optimal value when the choice is based on ranks. It is known that if $\mathcal{F}$ is the family of all normal distributions a minimax stopping rule gives asymptotic probability α of choosing the largest while if $\mathcal{F}$ is the family of all uniform distributions a minimax rule gives asymptotic value e^{-1}. This note considers a case intermediate to these extremes.

Let $\mathcal{F}$ be a family of continuous distribution functions and suppose n i.i.d $F \in \mathcal{F}$ random variables $X_1, \cdots, X_n$ are observed sequentially with the object of choosing the largest. After X_j has been observed it must be chosen (and the process terminated) or rejected (and the observations continued). No knowledge of the future is allowed, no recall of rejected observations is possible, and one observation must be selected.

Gilbert and Mosteller (1966) investigated the case in which $\mathcal{F}$ is a single distribution function. They named this the full information (hereafter F.I.) problem, and observed that

$$P(X_{\sigma(n)} = L_n)\downarrow\alpha \doteq .58 \qquad \text{as} \quad n \to \infty$$

for the optimal stopping rule $\sigma(n)$ and $L_n = \max\{X_1, \cdots, X_n\}$.

At an opposite extreme from the F.I. problem is the case in which $\mathcal{F}$ is the class of all continuous distribution functions. If the available stopping rules are restricted to those based only on the ranks of the observations, this problem is equivalent to the Secretary Problem. The solution is well known (see Dynkin and Yushkevich (1966) or Gilbert and Mosteller (1966)) as is the following asymptotic result: if for each n $\rho(n)$ is the optimal stopping rule based on ranks

$$P(X_{\rho(n)} = L_n)\downarrow e^{-1} \doteq .368 \qquad \text{as} \quad n \to \infty.$$

In keeping with the Gilbert and Mosteller terminology for the F.I. problem we call this the no information (hereafter N.I.) problem.

This note will concern itself with the intermediate case in which $\mathcal{F}$ is neither the class of all continuous cdf's nor a single such cdf. We call this the partial information (hereafter P.I.) problem.

Received January 1979; revised July 1979.

AMS 1970 *subject classifications.* Primary 62L15; secondary 62A05.

Key words and phrases. Optimal stopping, minimax rules, secretary problem, invariance.

Petruccelli (1978) derived sufficient conditions for the existence of invariant stopping rules $\nu(n) \leqslant n$ satisfying

$$\lim_{n\to\infty} P\big(X_{\nu(n)} = L_n\big) = \alpha \tag{1}$$

for the P.I. problem given by a location and/or scale parameter family $\mathfrak{F}$. In particular (1) holds if $\mathfrak{F}$ is the family of all normal distributions.

Furthermore given these conditions (1) holds for minimax stopping rules $\nu(n)$ since a version of the Hunt-Stein theorem (Kiefer 1957) insures that for the P.I. problem in which $\mathfrak{F}$ is a location and/or scale parameter family a best invariant rule is minimax.

However (1) does not hold for all location and/or scale parameter families $\mathfrak{F}$. In fact Samuels (1978b), extending the work of Stewart (1978), has shown that for the P.I. problem defined by the family of all uniform distributions a best invariant (hence minimax) rule is the N.I. rule based only on ranks.

In what follows we investigate a P.I. problem in which the probability of choosing the largest observation when using a minimax rule is asymptotically between the two extremes quoted above. We will prove the following

THEOREM. *Let* $\mathfrak{F} = \{F_\theta\}_{\theta\in\mathbf{R}}$ *where* F_θ *is the* cdf *of the* $U[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ *distribution. Let* $\tau(n)$ *be a best invariant (hence minimax) stopping rule for the* P.I. *problem of length n defined by* $\mathfrak{F}$. *Then*

$$\lim_{n\to\infty} P(X_{\tau(n)} = L_n) \doteq .43517. \tag{2}$$

Let $X_1, X_2, \cdots$ be i.i.d. $U[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ random variables. For each $n \geqslant 1$ let L_n be as above, $D_n = \min\{X_1, \cdots, X_n\}$ and $R_n = L_n - D_n$. I denotes the indicator function.

For the P.I. problem of the theorem a best invariant rule has been shown to be (Petruccelli 1978)

$$\tau(n) = \min\{n, \min_{k\geqslant 2}\{k: X_k = L_k, R_k > \lambda_{k,n}\}\}$$

where $\lambda_{k,n}$ depends only on $n-k$ and $1 > \lambda_{2,n} > \lambda_{3,n} > \cdots > \lambda_{n-1,n} = 0$. Further, $\lambda_{k,n}$ satisfies the equation

$$\Sigma_{i=1}^{n-k} i^{-1}\binom{n-k}{i}\int_{\lambda_{k,n}}^{1} u^{n-k-i}(1-u)^i\, du = \int_{\lambda_{k,n}}^{1} u^{n-k}\, du. \tag{3}$$

Using (3) we can prove

LEMMA. *Let* $\{\lambda_{k,n}\}_{k=2}^{n-1}$ *be the decision numbers defining* $\tau(n)$. *Then* $\lambda_{k,n} = 1 - c_k/(n-k)$ *where* $\lim_{k\to\infty} c_k = c \doteq 2.1198$.

We now prove the theorem. The proof is inspired by a technique used by Samuels (1978a) in the full information case. Assume $\theta = \frac{1}{2}$ and let

$$\sigma(n) = kI(X_k = L_n)$$
$$\delta(n) = kI(X_k = L_{\sigma(n)-1}).$$

Thus $\sigma(n)$ is the arrival time of the largest observation and $\delta(n)$ is the arrival time of the largest before time $\sigma(n)$. Note that $L_n = L_{\sigma(n)}$ and $L_{\sigma(n)-1} = L_{\delta(n)}$.

Now

$$\{X_{\tau(n)} = L_n\} = \{R_{\delta(n)} \leqslant \lambda_{\delta(n),n}, R_{\sigma(n)} > \lambda_{\sigma(n),n}\}$$

and

$$\begin{aligned}(4)\quad & P\big(R_{\delta(n)} \leqslant \lambda_{\delta(n),n}, R_{\sigma(n)} > \lambda_{\sigma(n),n} \,|\, L_{\sigma(n)}, L_{\sigma(n)-1}, \sigma(n), \delta(n)\big) \\ &= P\big(D_{\delta(n)} \geqslant L_{\delta(n)} - \lambda_{\delta(n),n}, D_{\sigma(n)} \\ &\quad < L_{\sigma(n)} - \lambda_{\sigma(n),n} \,|\, L_{\sigma(n)}, L_{\sigma(n)-1}, \sigma(n), \delta(n)\big) \\ &= \Big[\big(\lambda_{\delta(n),n}/L_{\sigma(n)-1}\big)^{\delta(n)-1} - \big(1 - (L_n - \lambda_{\sigma(n),n})/L_{\sigma(n)-1}\big)^{\sigma(n)-2}\Big] \\ &\quad \times I\big(L_{\sigma(n)-1} \geqslant \lambda_{\delta(n),n}\big) \\ &\quad + \Big[1 - \big(1 - (L_n - \lambda_{\sigma(n),n})/L_{\sigma(n)-1}\big)^{\sigma(n)-2}\Big] \\ &\quad \times I\big(L_{\delta(n)-1} - \lambda_{\delta(n),n} < 0 < L_n - \lambda_{\sigma(n),n}\big)\end{aligned}$$

since, given the vector $(L_{\sigma(n)}, L_{\sigma(n)-1}, \sigma(n), \delta(n))$, the observations $X_1, \cdots, X_{\delta(n)-1}$, $X_{\delta(n)+1}, \cdots, X_{\sigma(n)-1}$ are i.i.d. with a $U[0, L_{\sigma(n)-1}]$ distribution.

Let $Z_n = n(1 - L_n) = n(1 - L_{\sigma(n)})$;

$Y_n = (\sigma(n) - 1)(1 - L_{\sigma(n)-1}/L_n)$;

$V_n = \sigma(n)/n$;

$U_n = \delta(n)/(\sigma(n) - 1)$.

Then $(Z_n, Y_n, V_n, U_n) \to_{\mathfrak{D}} (Z, Y, V, U)$ where Z, Y, V, U are mutually independent, $Z, Y \sim \exp(1)$, $V, U \sim U[0, 1]$. Since

$$\begin{aligned}&P(R_{\delta(n)} \leqslant \lambda_{\delta(n),n}, R_{\sigma(n)} > \lambda_{\sigma(n),n} | L_{\sigma(n)}, L_{\sigma(n)-1}, \sigma(n), \delta(n)) \\ &\qquad = P(R_{\delta(n)} \leqslant \lambda_{\delta(n),n}, R_{\sigma(n)} > \lambda_{\sigma(n),n} | Z_n, Y_n, V_n, U_n)\end{aligned}$$

we may rewrite (4) in terms of (Z_n, Y_n, V_n, U_n). Then taking limits we have

$$\begin{aligned}P(X_{\tau(n)} = L_n | Z_n, Y_n, V_n, U_n) &\to_{\mathfrak{D}} \phi(Z, Y, V, U) \\ &= \big(e^{-(cUV/(1-UV))}e^{ZUV}e^{YU} - e^{ZV}e^{-(cV/(1-V))}\big) \\ &\quad \times I((1 - UV)(Z + Y/V) \leqslant c) + \big(1 - e^{ZV}e^{-(cV/(1-V))}\big) \\ &\quad \times I((1 - V)Z < c < (1 - UV)(Z + Y/V)).\end{aligned}$$

Thus

$$\begin{aligned}P(X_{\tau(n)} = L_n) &\to E\phi(Z, Y, V, U) \\ &= \Big[e^{-c} - c\textstyle\int_1^\infty (e^{-c\gamma}/\gamma)\, d\gamma\Big]\sum_{i=1}^\infty c^i/(i(i!)) \\ &\quad + (e^c - 1)\textstyle\int_1^\infty (e^{-c\gamma}/\gamma)\, d\gamma \\ &\doteq .43517. \qquad \square\end{aligned}$$

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WORCESTER POLYTECHNIC INSTITUTE
WORCESTER, MASSACHUSETTS 01609