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EXACT MOMENTS OF THE SAMPLE CROSS CORRELATIONS OF MULTIVARIATE AUTOREGRESSIVE MOVING AVERAGE TIME SERIES

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SUMMARY. In this paper we derive an analytic expression for the exact moments of sample cross correlations for multivariate autoregressive moving average (MARMA) time series. A method for numerical evaluation of these moments is described and an example given.

1. Introduction

The extension of the Box and Jenkins (1976) model building philosophy for univariate ARMA (p, q) time series to their multivariate equivalents has received considerable attention in the literature recently. Tiao and Box (1981) compare some early approaches proposed by Granger and Newbold (1977), Wallis (1977) and Chan and Wallis (1977), and put forward some easy-to-follow methods of their own, the most notable being their interpretations of sample cross correlations. Jenkins (1979) and Jenkins and Alavi (1981) propose a similar model building philosophy, noting that the interpretations of the sample cross correlations can be a difficult problem. Following the notation of Tiao and Box (1981), we define the vector MARMA time series

$$\boldsymbol{\Phi}_p(B)\boldsymbol{X}_t = \boldsymbol{\Theta}_q(B)\boldsymbol{a}_t \qquad \dots \quad (1.1)$$

where $\qquad \boldsymbol{\Phi}_p(B) = \boldsymbol{I}-\boldsymbol{\Phi}_1 B-\dots-\boldsymbol{\Phi}_p B^p$

and $\qquad \boldsymbol{\Theta}_q(B) = \boldsymbol{I}-\boldsymbol{\theta}_1 B-\dots-\boldsymbol{\theta}_q B^q$

are $(k\times k)$ matrices of autoregressive and moving average coefficients in the back-shift operator B, $\boldsymbol{X}_t$ is a vector stationary and invertible series (possibly from suitably differencing the original data) and $\boldsymbol{a}_t$ is vector white noise, assumed to be distributed as $N(\boldsymbol{0}, \boldsymbol{\Sigma})$.

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For the general model (1.1), the theoretical lag-l cross covariance matrix, $\mathbf{\Gamma}(l)$ can be obtained from

$$\mathbf{\Gamma}(l) = \{\gamma_{ij}(l)\} = \begin{cases} \sum_{s=l-r}^{l-1} \mathbf{\Gamma}(s)\mathbf{\Phi}'_{l-s} - \sum_{s=0}^{r-l} \boldsymbol{\psi}_s \mathbf{\Sigma}\, \boldsymbol{\theta}'_{s+l}, & l = 0, 1, \ldots, r \\ \sum_{s=1}^{r} \mathbf{\Gamma}(l-s)\mathbf{\Phi}'_s & l > r \end{cases} \qquad \ldots \quad (1.2)$$

where the $\boldsymbol{\psi}$'s are obtained using the relationship

$$\boldsymbol{\psi}(B) = \mathbf{\Phi}_p^{-1}(B)\mathbf{\Theta}_q(B) = \boldsymbol{I} + \boldsymbol{\psi}_1 B + \ldots, \qquad \ldots \quad (1.3)$$

$\boldsymbol{\theta}_0 = -\boldsymbol{I}$, $r = \max(p, q)$ and (i) if $p > q$, $\mathbf{\Phi}_{p+1} = \ldots = \mathbf{\Phi}_r = \mathbf{0}$, (ii) if $q < p$ then $\boldsymbol{\theta}_{q+1} = \ldots = \boldsymbol{\theta}_r = \mathbf{0}$.

If sample cross covariances are defined by the equation

$$C_{ij}(l) = \frac{1}{n} \sum_{t=1}^{n-l} (X_{it} - \bar{X}_i)(X_{j,t+l} - \bar{X}_j) \qquad \ldots \quad (1.4)$$

where n is the length of the series X_{it} and $\bar{X}_i$ its mean, the elements of the cross covariance matrix $\{\gamma_{ij}\}$ can be estimated by the C_{ij}. The sample correlation matrix function $\boldsymbol{R}_l$ may then be defined to have elements $r_{ij}(l)$, where

$$r_{ij}(l) = C_{ij}(l)/\sqrt{C_{ii}(0)C_{jj}(0)}$$

and $\sqrt{C_{ii}(0)}$ is the standard deviation of X_{it}. A referee has pointed out to us that other definitions of $r_{ij}(l)$ are possible. The numerical expressions we derive in Section 2 can be easily modified to take into account these alternative definitions.

Calculation of the theoretical cross covariance function given by (1.2) can be obtained from p matrices of autoregressive coefficients, q matrices of moving average coefficients and the variance-covariance matrix $\mathbf{\Sigma}$. An algorithm for doing this is given by Pate and Davies (1983). As in univariate time series analysis, where the sample correlation function and its statistical properties play a central role in model identification, the matrix $\boldsymbol{R}_l$, coupled with its sampling properties, will be an essential part of multivariate time series model identification.

The asymptotic distribution of $\boldsymbol{R}_l$ is well known (see, for example, Hannan, 1970, p. 228) but that knowledge is not necessarily directly usable in performing significance tests based on finite sample realisations. For example, in univariate time series, the work of Davies, Triggs and Newbold (1977), Ansley and Newbold (1979) and Davies and Newbold (1979, 1980) have shown that using significance tests based on asymptotic theory can be grossly sub-optimal when applied to finite samples of time series. We expect similar difficulties to arise in multivariate time series, if inferences from $\boldsymbol{R}_l$ are based on asymptotic theory. In Section 2 we provide a method for obtaining all finite sample moments of the elements of $\boldsymbol{R}_l$ so that, if necessary, asymptotic moment results could be checked for their applicability in finite sample applications. The asymptotic variance of the elements of $\boldsymbol{R}_l$ can be obtained from an approximate formula given by Bartlett (1946). For MMA(q) processes we have, to order $1/n$,

$$V(r_{ij}(l)) \approx \frac{1}{(n-l)}\left[1+2\sum_{h=1}^{p}\rho_{ii}(h)\rho_{jj}(h)\right],\ l > q \qquad \dots \quad (1.5)$$

where $\rho_{ij}(h)$ are the theoretical cross correlations of the series $\boldsymbol{X}_{it}$. Elements of the matrices $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, ..., would be compared with their approximate standard errors calculated from (1.5) by replacing the ρ_{ii} by r_{ii}.

The application of (1.5) for MMA(q) processes is thus particularly appealing. In Section 3 exact finite sample standard deviations of the elements of $\boldsymbol{R}_l$ are compared with those asymptotic values obtained from (1.5) in one particular case. (A general study into the applicability of (1.5) is beyond the scope of the present paper; we merely show how this could be done using exact finite sample moments.)

Also, in Section 3 we describe the more difficult problem of dealing with multivariate mixed processes, and suggest an approach which allows exact moments of the mixed process-sample cross-correlations to be calculated relatively easily.

2. Exact moments of sample cross-correlations of MMA (q) time series

We consider initially the MMA(q) process derived from (1.1) and for which $\boldsymbol{X}_t = \boldsymbol{\Theta}_q(B)\boldsymbol{a}_t$. For a sample size n we define $\boldsymbol{X} = [\boldsymbol{X}_n', \boldsymbol{X}_{n-1}', \dots, \boldsymbol{X}_1']'$ to be the $(kn\times 1)$ observation vector,

$$\boldsymbol{\Theta} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{\theta}_1 & -\boldsymbol{\theta}_2 \dots -\boldsymbol{\theta}_q & \boldsymbol{0} & \dots\dots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & -\boldsymbol{\theta}_1 \dots & & -\boldsymbol{\theta}_q & \\ \cdot & \cdot & & & \cdot & \\ \cdot & \quad\cdot & & & \quad\cdot & \\ & & \cdot & & & \cdot \\ \boldsymbol{0} & & \boldsymbol{I} & -\boldsymbol{\theta}_1 & \dots\dots\dots & -\boldsymbol{\theta}_q \end{bmatrix}$$

is the $(kn \times k(n+g))$ matrix of MA coefficient matrices.

Also let $\boldsymbol{a} = [\boldsymbol{a}_n', \boldsymbol{a}_{n-1}', \dots, \boldsymbol{a}_{1-q}']$ be the $(k(n+q) \times 1)$ vector of random shocks.

Since $\boldsymbol{a} \sim N(\mathbf{0}, \boldsymbol{I} \otimes \boldsymbol{\Sigma})$, where $\otimes$ denotes the Kronecker product, we see that $\boldsymbol{X}$ is multivariate normal,

$$(\boldsymbol{X} \sim N(\mathbf{0}, \boldsymbol{\Theta}(\boldsymbol{I} \otimes \boldsymbol{\Sigma})\boldsymbol{\Theta}')) \qquad \dots \quad (2.1)$$

If $\boldsymbol{W} = [(\boldsymbol{X}_n - \overline{\boldsymbol{X}})', (\boldsymbol{X}_{n-1} - \overline{\boldsymbol{X}}), \dots, (\boldsymbol{X}_2 - \overline{\boldsymbol{X}})']'$ then we can write $\boldsymbol{W} = \boldsymbol{V}\boldsymbol{X}$ where $\boldsymbol{V}$ is a $((n-1) \times n)$ matrix of $(k \times k)$ matrices $\boldsymbol{V}_{ij}$ with

$$\boldsymbol{V}_{ii} = (1-1/n)\boldsymbol{I}, \qquad i = 1, \dots, n-1$$

$$\boldsymbol{V}_{ij} = \boldsymbol{V}_{ji} = -\boldsymbol{I}/n, \quad i \neq j.$$

It follows that $\boldsymbol{W} \sim N(\boldsymbol{\Theta}, \boldsymbol{\Omega})$ where

$$\boldsymbol{\Omega} = \boldsymbol{V}\boldsymbol{\Theta}(\boldsymbol{I} \otimes \boldsymbol{\Sigma})\boldsymbol{\Theta}'\boldsymbol{V}'.$$

Our initial objective is to write the $C_{ij}(l)$ and $C_{ii}(0)$ defined by (1.4) as quadratic forms in $\boldsymbol{W}$.

Define :

(i) $\boldsymbol{I}_{ij}$ to be a $(k \times k)$ matrix with unity in the (i, j)-th position and zeros everywhere else;

(ii) for $l \geqslant 1$, $\boldsymbol{Q}_{ij}(l)$ to be an $(n \times n)$ matrix of $(k \times k)$ matrices $\boldsymbol{\psi}_{rs}$, which are zero matrices, except those $\boldsymbol{\psi}_{rs}$ on the l-th super and subdiagonal of matrices which are $\boldsymbol{I}_{ji}$ and $\boldsymbol{I}_{ij}$ respectively

(iii) for $l = 0$, $\boldsymbol{Q}_{ij}(0)$ to be as in (ii) except that;

$$\boldsymbol{\psi}_{rs} = \begin{cases} \mathbf{0}, & r \neq s \\ \boldsymbol{I}_{ij} + \boldsymbol{I}_{ji}, & r = s \end{cases}$$

(iv) $\boldsymbol{\Delta}$ to be the $(kn \times k(n-1))$ matrix

$$\begin{bmatrix} & \boldsymbol{I} & & \\ -\boldsymbol{I} & -\boldsymbol{I} & \dots & -\boldsymbol{I} \end{bmatrix}$$

It follows that (1.4) may be written in the quadratic form

$$C_{ij}(l) = \boldsymbol{W}'\boldsymbol{\Delta}'\boldsymbol{Q}_{ij}(l)\boldsymbol{\Delta}\boldsymbol{W} \begin{cases} l \geqslant 0 \\ i, j = 1, 2, \dots, k. \end{cases}$$

The intermediate step of computing the vector $\boldsymbol{W}$ is used to avoid the singular covariance matrix of $[(\boldsymbol{X}_n-\bar{\boldsymbol{X}})', (\boldsymbol{X}_{n-1}-\bar{\boldsymbol{X}})', ..., (\boldsymbol{X}_1-\bar{\boldsymbol{X}})']'$. Using well known multivariate theory, the joint moment generating function (mgf) of $C_{ij}(l)$, $C_{ii}(0)$ and $C_{jj}(0)$ is

$$\begin{aligned}\phi(t_1, t_2, t_3) &= E[\exp\{t_1C_{ij}(l)+t_2C_{ii}(0)+t_3C_{jj}(0)\}]\\ &= |\boldsymbol{\Omega}|^{-1/2}|\boldsymbol{R}|^{-1/2} \qquad \ldots \quad (2.2)\end{aligned}$$

where $$\boldsymbol{\sigma}_{ij}(l) = \boldsymbol{\Delta}'\boldsymbol{Q}_{ij}(l)\boldsymbol{\Delta} \begin{cases} l \geqslant 0 \\ i, j = 1, ..., k\end{cases}$$

and $$\boldsymbol{R} = \boldsymbol{R}(t_1, t_2, t_3) = \boldsymbol{\Omega}^{-1}-2t_1\boldsymbol{\sigma}_{ij}(l)-2t_2\boldsymbol{\sigma}_{ii}(0)-2t_3\boldsymbol{\sigma}_{jj}(0).$$

Using this result and a generalisation of a theorem of Sawa (1972, 1978) proved in Appendix 1, we obtain a formula for all moments of $r_{ij}(l)$ ($l > 0$; $i, j = 1, 2, ..., k$). There is it shown that

$$E[(r_{ij}(l))^m] = \{\Gamma(m/2)\}^{-2}|\boldsymbol{\Omega}|^{-1/2}\int_0^\infty\int_0^\infty t_2^{m/2-1}\, t_3^{m/2-1}\left\{\frac{\partial^m}{\partial t_1^m}\,|\boldsymbol{R}|^{-1/2}_{t_1=0}\right\}dt_2dt_3 \qquad \ldots \quad (2.3)$$

where $\boldsymbol{R} = \boldsymbol{R}(t_1, -t_2, -t_3)$.

From a result by De-Gooijer (1980) we can write

$$\frac{\partial^m}{\partial t_1^m}\,|\boldsymbol{R}|^{-1/2} = \frac{(m-1)!}{2}\sum_{j=0}^{m-1}\frac{1}{(j!)}\,\frac{\partial^j}{\partial t_1^j}\,|\boldsymbol{R}|^{-1/2}2^{r-j}\mathrm{tr}\{(\boldsymbol{R}^{-1}\boldsymbol{\sigma}_{ij}(l))^{m-j}\}$$

which may be substituted in (2.3) to give our desired moments.

After substituting $m = 1, 2$ in (2.3) and simplifying, we find the first two moments to be

$$\begin{aligned}E[r_{ij}(l)] = \frac{1}{\pi}\,|\boldsymbol{\Omega}|^{-1/2}\int_0^\infty\int_0^\infty t_2^{-1/2}t_3^{-1/2}|\boldsymbol{R}(0, t_2, t_3)|^{-1/2}\\ \times\,\mathrm{tr}\{(\boldsymbol{R}(0, t_2, t_3))^{-1}\boldsymbol{\sigma}_{ij}(l)\}dt_2\,dt_3 \qquad \ldots \quad (2.4)\end{aligned}$$

$$\begin{aligned}E[(r_{ij}(l))^2] = |\boldsymbol{\Omega}|^{-1/2}\int_0^\infty\int_0^\infty |\boldsymbol{R}(0, t_2, t_3)|^{-1/2}\\ \times[(\mathrm{tr}\{(\boldsymbol{R}(0, t_2, t_3))^{-1}\boldsymbol{\sigma}_{ij}(l))\})^2+2\mathrm{tr}\{((\boldsymbol{R}(0, t_2, t_3))^{-1}\boldsymbol{\sigma}_{ij}(l))^2\}]dt_2dt_3 \qquad \ldots \quad (2.5)\end{aligned}$$

In the univariate case, Ali (1984) has obtained a simplification of the second moment formula given by De-Gooijer (1980). However, our bivariate formula (2.5) does not lend itself readily to the approach suggested by Ali.

3. COMPUTATION OF EXACT MOMENTS OF MARMA TIME SERIES

In practice it will only be of interest to compute (2.3) for small values of m. Consequently, in Appendix 2 we present an efficient numerical integration procedure to evaluate (2.4) and (2.5) to give the first two exact moments of the sample cross correlations for MMA(q) time series at all lags.

In the case of computing the exact moments of sample cross correlations for mixed MARMA (p, q) series there are two approaches possible.

The first is to use results of Nicholls and Hall (1979), who obtained a closed form expression for the covariance matrix of a MARMA (p, q) process. Their approach involves obtaining an expression for the covariance matrix for the q pre-period noise values and p pre-period observation values in terms of $\boldsymbol{\Phi}$, $\boldsymbol{\Theta}_q$ and $\boldsymbol{\Sigma}$. This expression can be used to deduce the elements of the desired covariance matrix, $\boldsymbol{C}$, say, (see Nicholls and Hall's equation 16) in terms of $\boldsymbol{\Phi}_p$, $\boldsymbol{\Theta}_q$ and $\boldsymbol{\Sigma}$. We note that this matrix is extremely complicated and is very difficult to visualise. Nevertheless, we can replace the covariance matrix for the pure MMA(q) process, $\boldsymbol{\Theta}(\boldsymbol{I} \otimes \boldsymbol{\Sigma})\boldsymbol{\Theta}'$ in (2.1), by $\boldsymbol{C}$ and carry out the procedures described in Section 2 to obtain an equivalent formula to (2.3) for MARMA(p, q) processes.

The second approach, which is the one we adopt, is to express the MARMA(p, q) process as a long MMA time series. If we cut off the infinite MMA polynomial defined by (1.3) at some order q^*, which is assumed to be "large", the approach adopted in Section 2 may be applied to the appropriate MMA(q^*) representation of the original MARMA(p, q) process.

In fact the infinite MA approach was used by De-Gooijer (1980) in the univariate case. Evidently he opted for this owing to the relative simplicity in the calculations when compared with using the general results of Newbold (1974).

Table 3.1 contains the computed exact mean and standard deviation of the sample cross correlation matrices for a MMA(1) process with

$$\boldsymbol{\theta}_1 = \begin{bmatrix} 0{\cdot}2 & 0{\cdot}3 \\ -0{\cdot}6 & 1{\cdot}1 \end{bmatrix}, \qquad \boldsymbol{\Sigma} = \begin{bmatrix} 4 & 1 \\ 1 & 1 \end{bmatrix},$$

for lags $l = 0, 1, 2$ and sample sizes $n = 10, 25, 50$ and ∞. (The sample sizes are chosen for illustrative purposes only.)

The cross correlation matrices at lags 0 and 1, for $n = \infty$, were obtained from the solution of (1.2) using an algorithm described by Pate and Davies (1983). For comparison, we also evaluated the Bartlett (1946) formula (1.5) for approximate variances of sample cross correlations, to obtain standard deviations for lags $l \geqslant 2$ (in practice computed sample cross corelations would be substituted in (1.5), whereas we have used the exact theoretical cross correlations from the algorithm of Pate and Davies (1983)).

Two interesting features emerge from Table 3.1.

(i) For this particular MMA process, the exact mean sample cross correlations for $l = 0, 1$ are remarkably close for all sample sizes considered. Examination of this phenomenon for other MMA time series seems justified. This could be done using the exact moment computation as described in this paper or via a simulation study.

(ii) As is to be expected the approximate standard deviations of the sample cross correlations, derived from (1.5), are overestimates of the exact standard deviations. For $n = 50$ this bias is small and it may be worthwhile to investigate further other MMA time series for which this occurs.

For general MARMA time series we considered the bivariate processes B and C of Jenkins and Alavi (1981, p. 9). The former is MAR(1) while the latter is MARMA(1, 1). In each case we found that, as long as exact sample cross correlations were required only for lags below 10, the order q^* of the MMA(q^*) representation, need not be more than 10. As is to be expected the bias in estimation for these two processes was more severe for small sample sizes than in the pure MMA(1) process reported in Table 3.1.

Further results using these techniques are available from the authors in a technical report. The computer program to evaluate these moments can be obtained by writing to the first named author.

4. Conclusions

In this paper we have derived an analytic expression for all finite moments of sample cross correlations of MMA(q) time series and shown how these can be evaluated. By expressing general MARMA(p, q) processes as long MMA series, the methods are directly applicable to such mixed processes.

The evaluations of these moments involve a vast amount of numerical integration of functions of products of matrices, which on conventional computers, can take a long time. A further saving in time could be obtained by making more use of the matrix structures for the particular process being considered and/or using, where available, a distributed array processor.

TABLE 3.1. EXACT MEAN AND STANDARD DEVIATION OF SAMPLE CROSS CORRELATION FUNCTION FOR A BIVARIATE MA(1) PROCESS

	LAG $l = 0$		$l = 1$		$l = 2$		BARTLETT $l \geqslant 2$
	MEAN	ST DEV	MEAN	ST DEV	MEAN	ST DEV	ST DEV
$n = 10$	$\begin{bmatrix} 1 & 0.243 \\ 0.243 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.318 \\ 0.318 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.256 & 0.332 \\ -0.182 & -0.234 \end{bmatrix}$	$\begin{bmatrix} 0.255 & 0.308 \\ 0.307 & 0.256 \end{bmatrix}$	$\begin{bmatrix} -0.056 & -0.038 \\ -0.046 & -0.060 \end{bmatrix}$	$\begin{bmatrix} 0.273 & 0.297 \\ 0.301 & 0.269 \end{bmatrix}$	$\begin{bmatrix} 0.336 & 0.333 \\ 0.333 & 0.331 \end{bmatrix}$
$n = 25$	$\begin{bmatrix} 1 & 0.265 \\ 0.265 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.205 \\ 0.205 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.251 & 0.379 \\ -0.168 & -0.220 \end{bmatrix}$	$\begin{bmatrix} 0.188 & 0.212 \\ 0.205 & 0.183 \end{bmatrix}$	$\begin{bmatrix} -0.024 & -0.014 \\ -0.022 & -0.025 \end{bmatrix}$	$\begin{bmatrix} 0.20 & 0.205 \\ 0.206 & 0.195 \end{bmatrix}$	$\begin{bmatrix} 0.212 & 0.211 \\ 0.211 & 0.209 \end{bmatrix}$
$n = 50$	$\begin{bmatrix} 1 & 0.27 \\ 0.27 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0.171 \\ 0.171 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.251 & 0.394 \\ -0.162 & -0.217 \end{bmatrix}$	$\begin{bmatrix} 0.160 & 0.185 \\ 0.156 & 0.143 \end{bmatrix}$	$\begin{bmatrix} -0.012 & -0.070 \\ -0.012 & -0.013 \end{bmatrix}$	$\begin{bmatrix} 0.152 & 0.152 \\ 0.152 & 0.152 \end{bmatrix}$	$\begin{bmatrix} 0.150 & 0.149 \\ 0.149 & 0.148 \end{bmatrix}$
$n =$	$\begin{bmatrix} 1 & 0.279 \\ 0.279 & 1 \end{bmatrix}$		$\begin{bmatrix} -0.252 & 0.407 \\ -0.157 & -0.215 \end{bmatrix}$		$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$		

Appendix 1

MOMENTS OF THE SAMPLE CROSS CORRELATIONS

A generalisation of Sawa's lemma (Sawa, 1972 and 1978) is as follows: Let Q_1, Q_2 and Q_3 be random variables with $Q_2, Q_3 \geqslant 0$ almost surely. Then if the joint mgf of Q_1, Q_2, Q_3, $\phi(t_1, t_2, t_3)$ is defined for $|t_1| < \varepsilon$ some $\varepsilon > 0$, and $t_2, t_3 < 0$, and if $E[(Q_1/\sqrt{Q_2Q_3})^m]$ exists, then

$$E[(Q_1/\sqrt{Q_2Q_3})^m] = \Gamma^{-2}(m/2) \int_0^\infty \int_0^\infty t_2^{m/2-1}\, t_3^{m/2-1}\, \phi(t_2, t_3)dt_2\, dt_3$$

where $$\phi(t_2, t_3) = \frac{\partial^m}{\partial t_1^m}\, \phi(t_1, -t_2, -t_3)|_{t_1=0}.$$

Proof: By Fubini's theorem,

$$E[(Q_1/\sqrt{Q_2Q_3})^m]$$

$$= E\left[Q_1^m \left\{\Gamma^{-1}(m/2) \int_0^\infty t_2^{m/2-1}\, e^{-Q_2t_2}dt_2\right\} \left\{\Gamma^{-1}(m/2) \int_0^\infty t_3^{m/2-1}\, e^{-Q_3t_3}\, dt_3\right\}\right]$$

$$= \Gamma^{-2}(m/2) \int_0^\infty \int_0^\infty t_2^{m/2-1}\, t_3^{m/2-1}\, E[Q_1^m\, e^{-Q_2t_2}\, e^{-Q_3t_3}]dt_2dt_3.$$

From the properties of the m.g.f.

$$E[Q_1^m e^{-Q_2t_2}\, e^{-Q_3t_3}] = \frac{\partial^m}{\partial t_1^m}\, \phi(t_1, -t_2, -t_3)|_{t_1=0}$$

which gives the desired result.

Applying this result after substituting Q_1, Q_2 and Q_3 by $C_{ij}(l)$, $C_{ii}(0)$ and $C_{jj}(0)$ respectively, and employing (2.2), we obtain

$$E[(r_{ij}(l))^m] = \Gamma^{-2}(m/2)\,|\,\Omega\,|^{-\frac{1}{2}} \int_0^\infty \int_0^\infty t_2^{m/2-1}\, t_3^{m/2-1}$$

$$\left[\frac{\partial^m}{\partial t_1^m}\, \left|\, \Omega^{-1} - 2t_1\sigma_{ij}(l) + 2t_2\sigma_{ii}(0) + 2t_3\sigma_{jj}(0)\right|^{-\frac{1}{2}}\right]_{t_1=0} dt_2dt_3$$

Appendix 2

DETAILS OF THE NUMERICAL INTEGRATION

Numerical evaluation of integrals on the right hand side of (2.4) and (2.5) over the first quadrant $[0, \infty)\times[0, \infty)$ can be simplified by a bilinear transformation into the square $(-1, 1]\times(-1, 1]$.

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The integrals required are of the form

$$I_1 = D(i, j, l) = K_1 \int_0^\infty \int_0^\infty \frac{1}{\sqrt{uv}} \frac{\operatorname{tr}(\boldsymbol{R}^{-1}\boldsymbol{\sigma}_{ij}(l))}{\sqrt{|\boldsymbol{R}|}}\, dudv$$

$$I_2 = G(i, j, l) = K_2 \int_0^\infty \int_0^\infty \frac{1}{\sqrt{|R|}} \{[\operatorname{tr}(\boldsymbol{R}^{-1}\boldsymbol{\sigma}_{ij}(l))]^2 + 2\operatorname{tr}(\boldsymbol{R}^{-1}\boldsymbol{\sigma}_{ij}(l))^2\}\, dudv$$

where $K_1 = 1/(\pi\sqrt{|\boldsymbol{\Omega}|})$; $K_2 = 1\sqrt{/|\boldsymbol{\Omega}|}$

and $\boldsymbol{R} = \boldsymbol{\Omega}^{-1} + 2u\boldsymbol{\sigma}_{ii}(0) + 2v\boldsymbol{\sigma}_{jj}(0).$

We illustrate the procedure by reference to I_1.

The main problem is the efficient formation of $\boldsymbol{R}^{-1}\boldsymbol{\sigma}$ many times in the (u, v) plane. There are two difficulties associated with the integrations :

(a) a decision has to be made on how large an area should be taken in the (u, v) plane over which the integrals are to be evaluated,

(b) the number of points of evaluation of the integrands in the (u, v) plane.

If we apply the bilinear transformations,

$$u = \frac{1-a}{1+a}; \qquad v = \frac{1-b}{1+b}$$

then I_1, which is of the form

$$\int_0^\infty \int_0^\infty \frac{1}{\sqrt{uv}} f(u, v)\, dudv,$$

immediately becomes

$$4 \int_{-1}^1 \int_{-1}^1 \frac{1}{\sqrt{1-a^2}\sqrt{1-b^2}}\, g(a, b)\, dadb$$

where $$g(a, b) = \frac{1}{(1+a)(1+b)} f\left(\frac{1-a}{1+a}, \frac{1-b}{1+b}\right).$$

The factors $(1-a^2)^{-\frac{1}{2}}(1-b^2)^{-\frac{1}{2}}$ suggest that Gauss Chebyschev quadrature is appropriate, and one can derive a quadrature for I_1 as

$$I_1 \approx 4\frac{\pi^2}{m^2} \sum_{i=1}^m \sum_{j=1}^m g(a_i, b_j)$$

where $$a_i = \cos[(2i-1)\pi/(2m)]$$
$$b_j = \cos[(2j-1)\pi/(2m)]$$

and m is the number of points chosen in each direction in the a, b square. Remarkably, the non-uniform spacing of these points is beneficial to the problem at hand, since they are more clustered at the ends of the interval $[-1, 1]$. When $a = 1$ or $b = 1$, we have a singularity in I_1 and so more points will be needed there. As a and b tend to -1 we are rapidly approaching ∞ in the (u, v) plane so that many of the points (a_i, b_j) can be ignored in the double summation.

For choice of m, trial and error was needed. We found in the moving average case that for I_1, $m = 36$ gave a relative accuracy of 10^{-3}, whereas the same accuracy in I_2 was not achieved until $m = 42$. A possible reason for the latter result is that I_2 lacks the factor $1/\sqrt{uv}$ and so converges more slowly as $u, v \to \infty$.

We note that in I_1 and I_2 the formation of $\boldsymbol{P} = \boldsymbol{R}^{-1}\boldsymbol{\sigma}$ takes a time proportional to the cube of the order $\boldsymbol{P}$ like the evaluation of $\boldsymbol{R}^{-1}$ itself. However, since it is only $\operatorname{tr}\boldsymbol{P}$ and $\operatorname{tr}\boldsymbol{P}^2$ that are required, we note that these values may be found by direct formation of $\boldsymbol{R}^{-1}\boldsymbol{\sigma}$, in column order, by a forward and backward substitution in each column of $\boldsymbol{\sigma}$. We solve the equations $\boldsymbol{RP} = \boldsymbol{\sigma}$ having first found the Cholesky factorisation $\boldsymbol{R} = \boldsymbol{LL}'$ and obtain $1/\sqrt{|\boldsymbol{R}|}$ as a by product of the $\boldsymbol{L}$-factor. Obtaining $\operatorname{tr}\boldsymbol{P}$ and $\operatorname{tr}\boldsymbol{P}^2$ is now an easy matter, the time taken for the latter being proportional to the square of the order of the matrix since we only need diagonal elements of $\boldsymbol{P}\times\boldsymbol{P}$.

Program timings. The picture is confused by the vast amount of subsidiary array processing, which is more important for small n.

The Fortran compiler's optimising switch produced about a 15% saving in time for large n, while *fixing* $m = 42$ (rather than allowing the program to decide the optimal m) produced 50% savings. A further 22% saving was achieved by forming $\boldsymbol{R}^{-1}\boldsymbol{\sigma}$ directly and a further 25% was saved by exploiting the u—v symmetry in the integrands in the case $i = j$. (Each percentage being on the then-current run-times.)

On a DEC 2060, with $n = 25$ and $k = 2$, each integral now takes on average 10 mins. of CPU time (7 for the symmetric cases, 14 otherwise). If this time seems excessive, we should merely point out that to the required degree of accuracy (3rd d.p.) the evaluation of $\boldsymbol{R}^{-1}\boldsymbol{\sigma}$ is needed 1450 times for each (i, j, l).

Some useful further savings could be made by taking into account the sparsity structure of the $\boldsymbol{\sigma}_{ij}(l)$ matrices which in certain cases have half their columns as null vectors. This would considerably reduce the formation time for $\boldsymbol{R}^{-1}\boldsymbol{\sigma}_{ij}(l)$.

Given a distributed array processor (DAP) with $\geqslant 42 \times 42$ array of CPU's, and sufficient memory, it should be possible to compute the integrand values at all points simultaneously, thereby reducing the time by a factor of $\sim (42)^2$.

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