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SOME RESTRICTIONS ON THE USE OF CORNER METHOD HYPOTHESIS TESTS

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ABSTRACT

Beguin et al (1980) introduced the Corner Method as a tool for identifying the order (p,q) of an ARMA process. In addition they derived approximate hypothesis tests, based on asymptotic theory, to aid in the identification. We show that there are restrictions implicit in the use of these tests which, if violated, could yield spurious results.

1. INTRODUCTION

In some recent papers by Gray, Kelley and McIntire (1978), Beguin, Gourieroux and Monfort (1980), Woodward and Gray (1981) Jenkins and Alavi (1981) and Glaseby (1982), there has been a great deal of interest in finding simple criteria, based on sam-

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ple autocorrelations, with which to identify the autoregressive and moving average orders of an ARMA (p,q) process.

In this note we consider some aspects of the Corner Method criterion of Beguin et al (1980). This method is primarily exploratory, relying on recognition of a pattern in arrays of functions of sample autocorrelations. However, these authors also developed statistical tests for such patterns based on asymptotic theory. These are described in Section 2.

In Section 3 we show that there are restrictions implicit in the use of these tests. These restrictions are not obvious a priori and are not mentioned by Beguin et al (1980). However, by violating these restrictions the user of these tests runs a high risk of obtaining spurious results.

2. STATISTICAL TESTS FOR PATTERNS IN CORNER METHOD ARRAYS

We assume the stochastic process $\{X_t: t \in Z\}$ follows the ARMA(p,q) model (Box and Jenkins, 1976)

$$\phi(B)X_t = \theta(B)a_t \tag{2.1}$$

where

$$\phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i$$

$$\theta(B) = 1 - \sum_{i=1}^{q} \theta_i B^i \quad ,$$

B is the backshift operator and $\{a_t: t=0,\pm1,\pm2,\ldots\}$ is a white noise process with zero mean and covariance and with constant variance σ^2.

Assume also that the polynomials $\phi(z)$ and $\theta(z)$ have no zeros in common and that all their roots lie outside the unit circle.

Define the autocovariance and autocorrelation functions of the process to be, respectively,

$$\gamma(h) = E(X_t X_{t+h}),\ \rho(h) = \gamma(h)/\gamma(0),\ h \in Z.$$

For a realization $x_1,\ldots,x_T$ from this process define the sample autocorrelation function to be

$$r(h) = \sum_{t=1}^{T-h} (x_t-\bar{x})(x_{t+h}-\bar{x}) / \sum_{t=1}^{T} (x_t-\bar{x})^2, \quad h = 0,1,\ldots$$

where $\bar{x} = T^{-1} \sum_{t=1}^{T} x_t$.

In the sequel $r(h)$ will provide the point estimate of $\rho(h)$ and $\rho(-h)$. Further, if

$$V = f(\rho(i_1),\rho(i_2),\ldots)$$

is a function of the theoretical autocorrelations then we will denote its sample analogue, obtained by replacing the ρ's with the corresponding sample estimates r, as

$$\hat{V} = f(r(i_1),r(i_2),\ldots,).$$

For each $i \geq 0$, $j \geq i$ define the $j\times j$ matrix

$$B(i,j) = \begin{bmatrix} \rho(i) & \rho(i-1) & \ldots & \rho(i-j+2) & \rho(i-j+1) \\ \rho(i+1) & \rho(i) & \ldots & \rho(i-j+3) & \rho(i-j+2) \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \rho(i+j-1) & \rho(i+j-2) & \ldots & \rho(i+1) & \rho(i) \end{bmatrix},$$

and denote the determinant of $B(i,j)$ by $\Delta(i,j)$.

By replacing the ρ's in $B(i,j)$ by the sample quantities r we obtain the sample analogue $\hat{\Delta}$ of Δ.

The Corner Method is based on the fact that

$$\Delta(i,j) \equiv 0, \quad i \geq q+1,\ j \geq p+1$$
$$\neq 0, \quad \text{otherwise.}$$

One uses this fact to search for a pattern among the computed $\hat{\Delta}(i,j)$, displayed in matrix form, which will indicate the true values of p and q for the process (2.1)

As an aid in identifying such a pattern Beguin et al (1980) developed an approximate hypothesis test based on the asymptotic distribution of the $\hat{\Delta}$'s. While the test can be made more general we restrict our development to the null and alternative hypotheses of primary interest in confirming the pattern described above. We test

$$H_0: \Delta(i_\ell, j_\ell) = 0, \{(i_\ell, j_\ell), 1 \le \ell \le m\} \subset \{(i,j), i>0, j>0\} \quad (2.2)$$

against H_a: $\overline{H}_0$

where $\overline{H}_0$ is the complementary hypothesis that H_0 is not true.

Under H_0

$$\sqrt{T}(\hat{\Delta}(i_1, j_1), \ldots, \hat{\Delta}(i_m, j_m)) \xrightarrow{D} N(\underset{\sim}{0}, \Omega) \quad (2.3)$$

with $\Omega = HGH'$, where G is the $\eta \times \eta$ covariance matrix of $\rho(1), \ldots, \rho(\eta)$, $\eta = \max_{1 \le \ell \le m} (i_\ell + j_\ell - 1)$, and H is the $m \times \eta$ matrix $\{h_{k\ell}\}$ with elements

$$h_{k\ell} = \frac{\partial \Delta(i_k, j_k)}{\partial \rho(\ell)} .$$

In view of (2.3) Beguin et al (1980) suggested as a test of (2.2), computing

$$\chi^2 = T(\hat{\Delta}(i_1, j_1), \ldots, \hat{\Delta}(i_m, j_m)) \hat{\Omega}^{-1} (\hat{\Delta}(i_1, j_1), \ldots, \hat{\Delta}(i_m, j_m))'$$

and comparing with the critical value of the χ^2_m distribution.

3. RESTRICTIONS ON CORNER METHOD HYPOTHESIS TESTS

While the tests described in the last section appear to be completely general with respect to the set of indices $\{(i_\ell, j_\ell)\}_{\ell=1}^m$, there are in fact restrictions on the total number of indices permitted and on which specific indices may be used if the asymptotic distribution (2.3) is to have a nonsingular covariance matrix.

Clearly if this covariance matrix is singular the test statistic χ^2 will have unstable behavior for large T and unsatisfactory statistical properties for small and moderate T.

The restrictions we have found fall into two categories: those on the overall dimension or number of indices allowed and those on the matrix H of partial derivatives. In addition we have found a restriction for an ARMA(1,1) and a specific set of indices, which falls into neither of these categories.

3.1 Restrictions on Overall Dimension.

These restrictions arise from the simple fact that Ω is singular if $m > \eta$. For the Corner Method there are several configurations of the set of indices which it may be of interest to consider. Among these are:

i. Vertical or horizontal lines

By this configuration we mean the set of indices

$$\{(i_\ell,j_\ell)\}_{\ell=1}^{m} = \{(i,j_0)\}_{i=i_0}^{i_0+m-1} \;,\; \{(i_0,j)\}_{j=j_0}^{j_0+m-1} \quad \text{respectively}$$

for some i_0, j_0.

Then $\eta = i_0 + j_0 + m-2$ in each case and so Ω is singular if $i_0 + j_0 \le 1$. However, since we would be testing only indices $i_0 \ge 1$, $j_0 \ge 1$, there is no restriction of this type on our tests.

ii. Orthogonal arms

By this configuration we mean the set of indices

$$\{(i_\ell,j_\ell)\}_{\ell=1}^{m} = \{(i_0,j)\}_{j=j_0}^{j_1} \cup \{(i,j_0)\}_{i=i_0+1}^{i_1} \;,$$

some $i_0 < i_1$, $j_0 \le j_1$.

Then $m = (i_1 - i_0) + (j_1 - j_0) + 1$

$\eta = \max\{j_1+i_0-1, i_1+j_0-1\}$,

and Ω is singular if $2i_0 < i_1-j_0+2$

and $2j_0<j_1-i_0+2$.

In the particular case that the arms are of equal length $d = i_1 - i_0 = j_1 - j_0$, Ω is singular if $i_0 + j_0 < d + 2$.

iii. Rectangles

By this configuration we mean $\{(i_\ell,j_\ell)\}_{\ell=1}^m=\{(i,j): i_0 \le i \le i_1, j_0 \le j \le j_1\}$, some $i_0 \le i_1$, $j_0 \le j_1$.

Then $m = (i_1 - i_0 + 1)(j_1 - j_0 + 1)$, $n = i_1 + j_1 - 1$ and Ω is singular if

$$(i_1 - i_0 + 1)(j_1 - j_0 + 1) > i_1 + j_1 - 1.$$

If the rectangle is a square with $d = i_1 - i_0 = j_1 - j_0$ then Ω is singular if
$d > \sqrt{i_0 + j_0 - 2}$.

3.2 Restrictions on the Matrix of Partial Derivatives.

Ω will also be singular if the matrix H of partial derivatives, is badly behaved. Such considerations lead us to.

Theorem 3.1. With regard to the hypothesis test (2.2), if $i_\ell \ge j_\ell + q - p$ and $j_\ell \ge p + 2$, for any $1 \le \ell \le m$, then Ω is singular.

The proof of this theorem follows directly from the following lemma whose proof is found in the appendix.

Lemma 3.1 For any ARMA(p,q) process $\frac{\partial\Delta(i,j)}{\partial\rho(k)} = 0$ for any $k \ge 1$ if $i \ge j + q - p$ and $j \ge p + 2$.

In terms of the configurations considered in §3.1 the restrictions of Theorem 3.1 will have no effect on tests using the vertical line configuration if $j_0 \le p + 1$ or on tests using the horizontal line configuration if $i_0 \le q + 1$. There will be no effect on tests using the orthogonal arm configuration if $j_0 \le p + 1$ and $i_0 \le q + 1$. If the rectangular configuration is used there will be no effect if $j_1 \le p + 1$ or if $i_1 \le q + 1$.

3.3 Another Restriction

The above are not the only possible conditions under which Ω will be singular as the following shows.

Consider an ARMA (1,1) process and suppose the configuration of the Corner Method test consists of orthogonal arms of length 1 with vertex at (2,2). That is, we test

H_0: $\Delta(2,2) = \Delta(3,2) = \Delta(2,3) = 0$

H_a: $\bar{H}_0$.

The results of §3.1 and §3.2 give no reason to expect Ω to be singular. Yet it can be shown that the 3×4 matrix H has rank 2 which implies that Ω is singular.

4. CONCLUSIONS

We feel that the restrictions on the use of the Corner Method hypothesis tests obtained in §3 may prove useful to those interested in the problem of identifying the orders p and q of an ARMA(p,q) process.

While no attempt has been made to exhaust the conditions under which the covariance matrix Ω can be singular, the final example discussed in §3 may have disturbing implications for the usefulness of these tests in their full generality.

5. APPENDIX

Proof of Lemma [illegible]: Write

$$B(i,j) = [b_i, b_{i-1}, \ldots, b_{i-j+1}]$$

where $b_m = [\rho(m), \rho(m+1), \ldots, \rho(m+j-1)]'$.

$\frac{\partial\Delta(i,j)}{\partial\rho(k)}$ is the sum of the cofactors of all entries $\rho(k)$ in $B(i,j)$. We will show that under the conditions of the lemma each cofactor must equal 0. To do so it suffices to show that any $j-1$ columns of $B(i,j)$ are linearly dependent.

We know from Beguin et al (1980) that

$$\underset{\sim}{b}_m - \sum_{\ell=1}^{p} \phi_\ell \underset{\sim}{b}_{m-\ell} = \underset{\sim}{0}\ , \quad m = i,i-1,\ldots,i-j+p+1. \tag{5.1}$$

Let B^ν be the matrix obtained by deleting column $\underset{\sim}{b}_\nu$ of $B(i,j)$. If $\nu < i-p$ $(\nu > i-j+p+1)$ then by (5.1)

$$\underset{\sim}{b}_i - \sum_{\ell=1}^{p} \phi_\ell \underset{\sim}{b}_{i-\ell} = \underset{\sim}{0} \quad \left(\underset{\sim}{b}_{\nu-1} - \sum_{\ell=1}^{p} \phi_\ell \underset{\sim}{b}_{\nu-1-\ell} = \underset{\sim}{0}\right)$$

proving that the columns of B^ν are linearly dependent.

If $i - p \leq \nu \leq i - j + p + 1$, consider the columns $\{\underset{\sim}{b}_\ell\}_{\ell=i-p-1}^{i}$. We know

$$\underset{\sim}{b}_i - \sum_{\ell=1}^{p} \phi_\ell\ \underset{\sim}{b}_{i-\ell-1} = \underset{\sim}{0}. \tag{5.2}$$

Assume $\phi_{i-\nu-1} \neq 0$. Then

$$\underset{\sim}{b}_\nu = \phi_{i-\nu-1}^{-1}\{\underset{\sim}{b}_{i-1} - \sum_{\substack{\ell=1 \\ \ell\neq i-\nu-1}}^{p} \phi_\ell \underset{\sim}{b}_{i-\ell-1}\}\ .$$

But this means

$$\underset{\sim}{0} = \underset{\sim}{b}_i - \sum_{\ell=1}^{p} \phi_\ell \underset{\sim}{b}_{i-\ell} = \underset{\sim}{b}_i - \sum_{\substack{\ell=1 \\ \ell\neq i-\nu}}^{p} \phi_\ell \underset{\sim}{b}_{i-\ell} - \phi_{i-\nu} \underset{\sim}{b}_\nu\ .$$

$$= \underset{\sim}{b}_i - \sum_{\substack{\ell=1 \\ \ell\neq i-\nu}}^{p} \phi_\ell \underset{\sim}{b}_{i-\ell} - \phi_{i-\nu}\phi_{i-\nu-1}^{-1}\underset{\sim}{b}_{i-1}$$

$$+ \phi_{i-\nu}\ \phi_{i-\nu-1}^{-1} \sum_{\substack{\ell=2 \\ \ell\neq i-\nu}}^{p+1} \phi_{\ell-1}\underset{\sim}{b}_{i-\ell}$$

$$= \underset{\sim}{b}_i - (\phi_1 + \phi_{i-\nu}\phi_{i-\nu-1}^{-1})\underset{\sim}{b}_{i-1} - \sum_{\substack{\ell=2 \\ \ell\neq i-\nu}}^{p} (\phi_\ell - \phi_{i-\nu}\phi_{i-\nu-1}^{-1}\phi_{\ell-1})\underset{\sim}{b}_{i-\ell}$$

$$+ \phi_{i-\nu}\phi^{-1}_{i-\nu-1}\phi_p b_{i-p-1}\,.$$

so that the columns of B^ν are linearly dependent.

If $\phi_{i-\nu-1} = 0$ then there is no contribution of b_ν to (5.2) and the columns of B^ν are linearly dependent.

BIBLIOGRAPHY

Beguin, J. M., Gourieroux, C. and Monfort A. (1980). Identification of a mixed autoregressive-moving average process: the corner method, Time Series, O. D. Anderson, ed., 423-436.

Box, G. and Jenkins, G. (1976). Time Series Analysis: Forecasting and Control, Holden Day.

Gray, H. L., Kelley, G. D. and McIntire, D. D. (1978). A new approach to ARMA modeling, Commun. Statist. Simula. Computa., B, 7, 1-77.

Glaseby, C. A. (1982). A generalization of partial autocorrelations useful in identifying ARMA models, Technometrics, 24, 3, 223-228.

Jenkins, G. M. and Alavi, A. S. (1981). Some aspects of modelling and forecasting multivariate time series, J. Time Series Analysis, 2, 1-47.

Woodward, W. A. and Gray, H. L. (1981). On the relationship between the S array and the Box-Jenkins Method of ARMA model identification, JASA, 76, 375, 579-587.

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