# On the Use of the General Partial Autocorrelation Function for Order Determination in ARMA($p$, $q$) Processes

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We show that the General Partial Autocorrelation Function (GPAC), which has recently been suggested to be used as one of a set of convenient tools for order identification in ARMA models, has unstable behavior when applied to time series of moderate length. Its use in detecting the order of MA components in real series is very limited and can only be recommended as a means to confirm a pure AR fit to the data.

KEY WORDS: Time series; Identification; Simulation; Empirical results.

## 1. INTRODUCTION

In recent papers, Gray, Kelley, and McIntire (1978) and Woodward and Gray (1981), referred to in this article as WG, suggest extensions of the usual partial autocorrelation function that is used to identify the univariate ARMA($p$, $q$) model

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = \epsilon_t - \theta_1 \epsilon_{t-1} - \cdots - \theta_q \epsilon_{t-q}$$

where $\epsilon_t$ is assumed to be iid white noise.

Let $\rho_j$ be the $j$th autocorrelation of the ARMA($p$, $q$) process. The $k$th partial autocorrelation function $\phi_{kk}^{(0)}$, is the last ($k$th) autoregressive coefficient in solving $k$ Yule-Walker equations. Using the WG notation we can write

$$\begin{aligned} \phi_{kk}^{(0)} &= \rho_1 \qquad k = 1 \\ &= \frac{|A(k, 0)|}{|B(k, 0)|} \text{ if } k > 1, \end{aligned}$$

where $B(s, t)$ is the ($s \times s$) matrix defined by

$$B(s, t) = \begin{pmatrix} \rho_t & \rho_{t-1} & \cdots & \rho_{t-s+1} \\ \rho_{t+1} & \rho_t & & \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \end{pmatrix},$$

and $A(s, t)$ is the matrix composed of the first $(s - 1)$ columns of $B(s, t)$ with the $s$th column given by $(\rho_{t+1}, \ldots, \rho_{t+s})'$. The general partial autocorrelation function (GPAC) is then defined by

$$\begin{aligned} \phi_{kk}^{(j)} &= \rho_{j+1}/\rho_j \qquad \text{if } k = 1 \\ &= |A(k, j)|/|B(k, j)| \quad \text{if } k > 1. \end{aligned} \tag{1.1}$$

WG advocate displaying the GPAC's in an array whose $(j, k)$th element is

$$\phi_{kk}^{(j)} (j = 0, 1, 2, \ldots; k = 1, 2, \ldots).$$

For such an array, where rows and columns are labeled as the MA and AR orders, respectively,

1. zero behavior occurs in the $q$th row for columns $p + 1, p + 2, \cdots$, and
2. constant behavior occurs in the $p$th column for rows $q, q + 1, \cdots$.

WG suggest substituting $r_j$, the sample autocorrelations for the observed process, for $\rho_j$ in (1.1) to help identify the orders $p$, $q$. They advocate a visual inspection of the array so obtained, bearing in mind the above population characteristics. No distributional results are given by WG, and they say little about how one should ascertain whether corresponding population quantities are constant or zero. Some of the estimates $\hat{\phi}_{kk}^{(j)}$ so obtained have been shown by Glaseby (1982) to have well-behaved asymptotic properties and satisfactory sampling properties in large samples. However, we believe the asymptotic properties in general are not straightforwardly usable in sample sizes likely to be found in practice.

From matrix theory we note that $B(k, 0)$ is positive definite, whereas $B(k, j)$ $(j > 0)$ is not necessarily so. Thus, the finite sampling distribution of the GPAC, based on (1.1), may be unstable, resulting in large standard deviations of the statistic $\hat{\phi}_{kk}^{(j)}$.

Our conjecture is not supported by simulation studies provided in a worked example by WG, but it is supported by evidence from Newbold and Bos (1983), where they showed that the finite sampling distribution of some of the statistics in the GPAC array has some undesirable properties. In this article we provide simulation evidence

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*Table 1. Empirical Mean and Standard Deviation of GPAC for Constant Behavior Column 3 for Various Sample Sizes of Process (2.1); 1,000 Simulations*

| Sample Size | MA Order 2 | | 3 | | 4 | | 5 | | 6 | | 7 | | 8 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | .275 | (.53) | .482 | (2.93) | .593 | (4.21) | .430 | (7.46) | .620 | (7.41) | .476 | (8.26) | 1.149 | (15.3) |
| 100 | .375 | (.25) | .280 | (4.98) | .446 | (3.51) | .395 | (8.31) | −.329 | (29.5) | .913 | (19.1) | .831 | (6.19) |
| 500 | .438 | (.10) | .419 | (.16) | .456 | (.45) | .426 | (.64) | .434 | (.94) | .382 | (2.96) | 1.670 | (64.1) |
| ∞ | .455 | | .455 | | .455 | | .455 | | .455 | | .455 | | .455 | |

NOTE: Figures within parentheses are standard deviations.

and some real data analyses to demonstrate difficulties with application of the whole GPAC array technique. In the next section we critically examine some of the examples cited by WG.

## 2. EVIDENCE IN FAVOR OF THE GPAC

Jenkins and Alavi (1981) proposed procedures for fitting vector autoregressive moving average time series models to data. In the univariate case, one of these procedures specializes to consider the behavior of the estimates $\hat{\phi}_{kk}^{(q)}$ ($k \geq 1$), generally termed the $k$th-order $q$-conditioned partial autocorrelations of the series. They therefore correspond to that row in the Woodward and Gray GPAC array for which there is zero behavior for $k > p$. Similar proposals are also mentioned by Tiao and Box (1981) for multivariate applications.

Newbold and Bos (1983) report some problems with using this statistic for identification purposes; in simulation studies for ARMA(1, 1) processes, they show that the empirical standard deviations of the 1-conditioned partial autocorrelations are large even for processes of 100 observations. Thus it would seem, even in this relatively simple process, the $\hat{\phi}_{kk}^{(1)}$ ($k \geq 2$) will have a heavy tailed distribution.

As an example demonstrating the usefulness of the whole of the GPAC array, WG simulated a single realization of length 300 from the ARMA(3, 2) process,

$$X_t - 1.5X_{t-1} + 1.21X_{t-2} - .455X_{t-3} = \epsilon_t + .2\epsilon_{t-1} + .9\epsilon_{t-2}, \quad (2.1)$$

and calculated the GPAC array for $p = 1, 2, \ldots, 5$ and $q = 0, 1, \ldots, 5$. They noted some problems with interpretation of the array, but those were not connected with sampling properties of the calculated statistics. Their Table 7 clearly identified an ARMA(3, 2) process from their realization. To investigate the possibility of poor sampling properties for the GPAC in rows that should contain zero behavior and columns that should exhibit constant behavior, series of length $n$ were generated from the ARMA(3, 2) model (2.1) for $n = 50, 100, 500$. For brevity we only include the means and standard deviations of the GPAC's relevant to identifying an ARMA(3, 2) model.

Table 1 shows the mean and standard deviations of the GPAC array for the column that should exhibit constant behavior for MA order $\geq 2$. Even when a sample size of 100 is considered, the empirical expected value of the GPAC does not behave in a constant way. The results are even more dramatic when the empirical standard deviations are examined. For sample size 500, the empirical means in Table 1 are reasonably constant for MA order below 6, but the corresponding standard deviations show that variability is still comparatively high.

Table 2 shows the mean and standard deviations of the GPAC array for the row that should exhibit zero behavior for AR order $\geq 3$. Note that the empirical means seem reasonably close to zero for sample sizes of 100 and 500, but the standard deviations are large even for a sample size of 100.

The instability of the standard deviations in Tables 1 and 2, together with similar results reported for ARMA(1, 1) processes by Newbold and Bos (1983), suggests strongly that the finite sampling distributions of the statistics in the GPAC array can have large variances.

WG advise caution in using the GPAC array to determine the MA order of the process, $q$, and comment that the constant behavior can occur before row $q$. As we

*Table 2. Empirical Mean and Standard Deviation of GPAC for Zero Behavior Row 2 for Various Sample Sizes of Process (2.1); 1,000 Simulations*

| Sample Size | AR Order 3 | | 4 | | 5 | | 6 | | 7 | | 8 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50 | .275 | (.53) | −.263 | (1.86) | .035 | (1.11) | −1.05 | (30.67) | .163 | (4.28) | .108 | (16.39) |
| 100 | .375 | (.25) | −.098 | (.30) | .102 | (.88) | −.085 | (3.24) | −.627 | (10.9) | .001 | (2.54) |
| 500 | .438 | (.10) | −.012 | (.11) | .012 | (.12) | .010 | (.11) | −.031 | (.22) | −.016 | (.12) |
| ∞ | .455 | | .0 | | .0 | | .0 | | .0 | | .0 | |

NOTE: Figures within parentheses are standard deviations.

show in the next section, our experience with simulated series of moderate length is that the constant or zero behavior often does not appear at all.

We encountered similar results in simulation studies of the other examples cited by WG. Based on these results, we suggest that the comforting picture created by single realization examples in WG should be viewed with caution. Further evidence for this is provided in Section 3, where we report some further simulation studies and our experiences with using the GPAC array technique on real data.

## 3. EXPERIENCE WITH APPLICATION OF THE GPAC ARRAY TECHNIQUE

In this section we report some of our experiences with using the GPAC technique on (a) series for which published analyses of mixed nonseasonal models already exist and (b) simulated series for mixed nonseasonal models.

For application to real data, we have excluded any series that are well established as being pure AR or pure MA. These types of models are readily identified using well-known techniques. We assumed that all series had been suitably differenced and/or deseasonalized before we applied the GPAC array technique, which is consistent with the suggestions of WG, in Section 4.

WG analyzed some real data reported by Makridakis (1978) and showed that the GPAC array technique could positively identify an ARMA(13, 1) model, which we note is close to the pure AR(13) model identified by Parzen (1979) using different methods. We also note that neither of these two models coincides with that originally identified by Makridakis.

Ozaki (1977) compared the Akaike AIC technique for automatic identification and estimation of ARMA processes with some previous "manual" analyses given by Box and Jenkins (1976). The analyses of series A, C, and E provided by Box and Jenkins (BJ) and Ozaki (AIC) appear to be noncontroversial and well-documented, and the data is readily available. For each series at least two identified models are entertained from either the manual BJ analyses or the AIC fitting procedure.

The application of the GPAC array technique to these series resulted only in agreement with BJ and/or AIC identifications for those series that could be assumed to be pure AR or very nearly so. The array technique did not suggest that a moving average term was present in any of the three series, even though both BJ and AIC suggest the possibility of such terms in two of the three series.

A real data series known to contain a heavy moving average component is series J, the gas furnace data, of Box and Jenkins (1976). These authors clearly identified and estimated a mixed ARMA(4, 2) model. Using the application of Akaike's AIC method suggested by Kitagawa (1977), the best fitting model was found to be ARMA(3, 2). Inspection of the appropriate rows and columns of the GPAC array for these data, corresponding to mixed orders suggested by both the former analyses, did not confirm either model as being appropriate. In fact, we could not identify any particular model by an overall inspection of the GPAC array. This is unfortunate because the series length is 296 and one might expect an encouraging picture to emerge from having such a large sample size.

To investigate the possibility that the GPAC array technique cannot help to identify series with moving average components, we simulated several mixed ARMA models and applied the method to single realizations.

We generated data from the ARMA(1, 1) model

$$X_t - \phi_1 X_{t-1} = \epsilon_t - \theta_1 \epsilon_{t-1} \qquad (\phi_1 = .4,\ \theta_1 = .7),$$

and the ARMA(1, 2) model

$$X_t - \phi_1 X_{t-1} = \epsilon_t - \theta_1 \epsilon_{t-1} - \theta_2 \epsilon_{t-2} \qquad (\phi_1 = .5,\ \theta_1 = -1.0,\ \theta_2 = .4),$$

for sample sizes 50, 100, and 200. No clear identification of an ARMA(1, 1) model was possible for sample sizes 50 and 100, and for sample size 200, only the zero-row behavior was reasonably clear. For the simulated ARMA(1, 2) model, the results were surprising and even more discouraging. We were unable to detect constant or zero behavior in the GPAC array for any of these sample sizes. The same was true on repeating the exercise for many realizations.

A referee has drawn our attention to certain refinements to improve constant and zero pattern recognition, for identification in the GPAC array, suggested by Parzen (1981). The second author believes, however, that in any case, one should expect "failure of such patterns to exist as most time series are not exactly ARMA processes." Although we agree with this sentiment to a certain extent, we feel justified in expecting some degree of success with real data that are established as having MA components and even more success with data that are generated from mixed ARMA models.

The complexity of the moving average component in models seems to be a crucial factor in determining whether the GPAC array technique can make a positive identification. The apparent success Woodward and Gray had in identifying the ARMA(3, 2) model given in their 1981 article seems to be the exception rather than the rule.

## 4. CONCLUSIONS

In this article we have shown that the sampling distribution of the GPAC can be ill-behaved, especially when applied to series with length likely to occur in practice. It has been suggested that the GPAC array can be a useful tool at the identification stage of ARMA modeling, especially when mixed ARMA time series are present and need to be identified. We find

1. that a positive identification can rarely be made for data that are well established to be mixed ARMA;

2. that when a positive identification can be made, it seems to agree only with previous analyses for pure AR (or near pure AR) identifications;

3. that the presence of MA terms in the model to be identified appears to dramatically alter the viability of the GPAC array technique, even for moderate sample sizes.

Thus, we feel that the GPAC array technique could be a useful tool for confirming the presence of pure autoregressive components for moderate sample sizes. When moving average terms are present, however, it is of limited use to identify them, even for large sample sizes.

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