Ma2201/CS2022 Final Exam 0000

Discrete Mathematics

D Term, 2019

Print Name: ____________________

Sign: ____________________

1	
2	
3	
4	
5	
6	
7	
8	
total	

Do any **6** of the following 8 problems. Clearly cross out the problems you do not want to be graded. We will grade the first six not blank, not crossed out problems.

1. (**5 points**) a) How many anagrams of 'alligator' are there?

♣ *Good old multiplicative principle, still giving you points:* $\frac{9!}{2!2!}$ *allowing for the over counting of the two l's and two a's.* ♣

b) How many anagrams of 'alligator' do not have all the vowels clumped together, like 'llaiaogtr'.

♣ *Well, how many DO? The number of ways to clump them is* $\frac{4!}{2!}$ *and then for each of those clumps, call it* V *for vowel, there are* $6!/2!$ *anagrams of* $llVgtr$. *So* $\frac{4!6!}{2!2!}$ *and we have to subtract these violators form all the* $\frac{9!}{2!2!}$ *and get*

$$\frac{9!}{2!2!} - \frac{4!6!}{2!2!} = \frac{9! - 4!6!}{2!2!}$$

unclumped anagrams. ♣

2. (**5 points**) Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$

a) Define an onto function $f : \mathcal{P}(A) \cup \mathcal{P}(B) \to \mathcal{P}(A) \cap \mathcal{P}(B)$.

♣ *Let's check the cardinalities first. Inclusion/Exclusion, or simple reasoning says* $|\mathcal{P}(A) \cup \mathcal{P}(B)| = |\mathcal{P}(A)| + |\mathcal{P}(B)| - |\mathcal{P}(A \cap B)| = 8 + 8 - 4 = 12$. *While the cardinality of the target is only* $|\mathcal{P}(A) \cap \mathcal{P}(B)| = |\mathcal{P}(A \cap B)| = 2^2 = 4$. *So there certainly are onto functions. Listing all twelve values is not so bad. But why not define* $f(S) = S \cap A \cap B$. *That is a function, and if* $S \subseteq A \cap B$, *then* $f(S) = S \cap A \cap B = S$, *so every element of* $\mathcal{P}(A \cap B)$ *is related at least once.* ♣

b) How many onto functions are there with that domain and target?

♣ *We already computed the domain and target cardinalities:*

$$\binom{4}{4}4^{12} - \binom{4}{3}3^{12} + \binom{4}{2}2^{12} - \binom{4}{1}2^{12} + \binom{4}{0}0^{12}$$

and I know some of you will think this is the easy *part!* ♣

3) (**5 points**) Prove by induction that for any natural number $n \geq 0$ that $n^3 - n$ is evenly divisible by 3.

Your prove *must* be clear, and complete.

♣ *We did the harder one about divisibility by* 6 *in class.*

Base Case: For $n = 0$ *the statement is that* $0^3 - 0$ *is divisible by* 3*, which it is because* $0 = 3 \cdot 0$.

Induction Step: Suppose for some n *that* $n^3 - n$ *is evenly divisible by* 3*, that is,* $n^3 - n = 3k$ *for some* $k \in \mathbb{N}$*. Then*

$$\begin{aligned}
(n+1)^3 - (n+1) &= (n^3 + 3n^2 + 3n + 1) - (n+1) \qquad \textit{by the binomial theorem} \\
&= (n^3 - n) + 3(n^2 + 3n) \\
&= 3k + 3(n^2 + 3n) \qquad \textit{by the induction hypothesis} \\
&= 3(k + n^2 + 3n)
\end{aligned}$$

and $k + n^2 + 3n \in \mathbb{N}$*. So* $(n+1)^3 - (n+1)$ *is also evenly divisible by* 3*, concluding the induction step.*

. *Therefore, the statement is true for all* $n \geq 0$ *by induction.* ♣

4) (**5 points**) Consider the following sentence:

For all positive reals x *and all positive reals* y*, there is a positive integer* m *and a positive integer* n *so that* $x^n = y^m$.

a) Write this in correct logical notation using no English words (or foreign languages either.)

♣ $\forall x \in \{x \in \mathbb{R} \mid x > 0\}, \forall y \in \{y \in \mathbb{R} \mid y > 0\}, \exists n \in \mathbb{N}, \exists m \in \mathbb{N}, [(n > 0) \wedge (m > 0) \wedge (x^n = y^m)]$ ♣

b) Write it's negation in correct logical notation, again using no English words (or foreign languages either, and without the symbol ¬.)

♣ $\exists x \in \{x \in \mathbb{R} \mid x > 0\}, \exists y \in \{y \in \mathbb{R} \mid y > 0\}, \forall n \in \mathbb{N}, \forall m \in \mathbb{N}, [(n = 0) \vee (m = 0) \vee (x^n \neq y^m)]$ ♣

5) (**5 points**) a) Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Define any predicate on $\mathcal{P}(A)$ which is true as often as it is false.

♣ *The domain has* $2^{10} = 1024$ *elements, so it would be best to describe it not using a list. We want a predicate which is true for half the subsets. We learned that half the subsets have even cardinality. So for* $S \in \mathcal{P}(A)$*, the predicate is: "The cardinality of* S *is even". Or if you want to write it more mathematically:*

$$\exists k \in \mathbb{N}, |S| = 2k$$

And that is true for half the subsets.

If you don't like that, half the sets have cardinality 4 *or less, and half have cardinality* 5 *or more, so* $[|S| \leq 4]$ *works, as well as* $[S \in \mathcal{P}_0(A) \cup \mathcal{P}_1(A) \cup \mathcal{P}_2(A) \cup \mathcal{P}_3(A) \cup \mathcal{P}_4(A)]$ ♣

b) Professor Carwasher claims that the sentence in problem 4 (the previous problem, not this one!) is a predicate and defines a Boolean function depending on two continuous and two discrete values. Professor Dishbreaker argues that the sentence is a simple statement, and that he wouldn't be surprised if it was false. Which one is correct? Why?

♣ *For once Professor Dishbreaker is correct.*

Since the values of x*, and* y *are universally quantified, and the values of* m *and* n *are existentially quantified, the statement does not depend any any variables at all, so is just a simple statement. And it is probably false too, but I don't know a proof.* ♣

6. (**5 points**) Consider that statement $(p \Rightarrow q) \Rightarrow (r \Rightarrow p)$.

a) Write the statement in conjunctive normal form.

♣ *Using the definition of implies:*

$$\begin{aligned}(p \Rightarrow q) \Rightarrow (r \Rightarrow p) &= (r \Rightarrow p) \vee \neg(p \Rightarrow q) \\ &= (p \vee \neg r) \vee \neg(q \vee \neg p) \\ &= (p \vee \neg r) \vee (\neg q \wedge p) \\ &= (p) \vee (\neg r) \vee (\neg q \wedge p)\end{aligned}$$

which is in disjunctive normal form, the solution to b.

Moreover, the third clause is unnecessary given the first, so it is equivalent to $(p \vee \neg r)$*, which is in conjunctive normal form with just one clause.*

If you want to get there more mechanically,

$$\begin{aligned}(p \vee \neg r) \vee (\neg q \wedge p) &= [(p \vee \neg r) \vee \neg q] \wedge [(p \vee \neg r) \vee p)] && \textit{distributive law} \\ &= [p \vee \neg r \vee \neg q] \wedge [p \vee \neg r \vee p] \\ &= [p \vee \neg r \vee \neg q] \wedge [p \vee \neg r]\end{aligned}$$

Which is in conjunctive normal form, and again, the first clause is unnecessary in the context of the second.

b is already done, and that leaves c. $(p \vee \neg r)$ *is an or statement, so is true* $3/4$ *of the values of* p *and* r*, the value of* q *being irrelevant.*

Or you can reason, the only way for $(p \vee \neg r)$ *to be false is if* $p = 0$*,* $r = 1$*, and* q *can be* 0 *or* 1*. So only* 2 *of the* 8 *values give false.* ♣

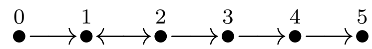
b) Write the statement in disjunctive normal form.

c) Considering the eight possible combinations of truth values for p, q and r, is it more likely that the statement is true, or false.

7. (**5 points**) Consider a relation $\square$ on $\{0,1,2,3,4,5\}$ and suppose $0\square 1$, $1\square 2$, $2\square 1$, $2\square 3$, $3\square 4$, and $4\square 5$.

a) Draw the relation diagram with the arrows corresponding the specified relationships, and all the consequent ones if $\square$ is a symmetric relation.

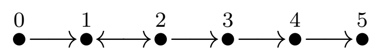
♣ *The new edges are in red, and just reverse the existing arrows.*



♣

b) Draw a second relation diagram showing the specified relationships, all consequent ones if $\square$, while not necessarily symmetric, is transitive.

♣ *The new edges are in green,*



Do you see where the loops come from? ♣

8. (**5 points**) How many many numbers at least 3 million and less than 9 million are divisible by 2, or 3 or 5?

Your answer should be in the form of an algebraic expression, with a justification which is clear and complete.

♣ *Let A be the set of subset of the* 6 *million consecutive numbers which are even, so* $|A| = 6 \cdot 10^6/2$.

Let B be the set of subset of the 6 *million consecutive numbers which divisible by* 3, *so* $|A| = 6 \cdot 10^6/3$.

Let C be the set of subset of the 6 *million consecutive numbers which divisible by* 5, *so* $|A| = 6 \cdot 10^6/5$.

We want to compute $A \cup B \cup C$ and use inclusion/exclusion.

The numbers in $A \cap B$ are divisible by 6. *So* $|A| = 6 \cdot 10^6/6$.

The numbers in $A \cap C$ are divisible by 10. *So* $|A| = 6 \cdot 10^6/10$.

The numbers in $B \cap C$ are divisible by 15. *So* $|A| = 6 \cdot 10^6/15$.

Lastly, the numbers in $A \cap B \cap C$ are divisible by 30. *So* $|A| = 6 \cdot 10^6/30$.

Thus

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ &= \frac{6 \cdot 10^6}{2} + \frac{6 \cdot 10^6}{3} + \frac{6 \cdot 10^6}{5} - \frac{6 \cdot 10^6}{6} - \frac{6 \cdot 10^6}{10} - \frac{6 \cdot 10^6}{15} + \frac{6 \cdot 10^6}{30} \end{aligned}$$

And that is all. ♣