Ma2201/CS2022 **Discrete Mathematics** D Term, 2018

Final Exam

Print Name: ____________________

Sign: ____________________

Please do any six of the following 8 problems. Each is worth 5 points. Cross out completely any problem not to be graded.

1. Suppose p and q are primes with $p > 5$ and $q > 11$, and suppose that $p \mid 3q$.

Label each of the following T for true, F for false, and X if it cannot be determined.

_____ $14q + 49p$ is prime.

_____ $p = q$.

_____ $3 \mid p$.

_____ $\gcd(3^p, q^3) = 1$.

_____ $p > 11$.

♣ *First is F since* $14q + 49p$ *factors as* $7(2q + 7p)$.

Second is T since $p \mid 3q$, *p must be one of the prime factors, of* $3q$, *since it isn't* 3, *it is too big, it must be q. That also says that* $p > 11$ *so fifth is also T*

Third is F since p is a prime not 3, so 3 *does not divide it.*

Fourth is T since 3^p *has only prime factors* 3, *and* q^3 *has only prime factors q.* ♣

2. In the land of Zork there are two kind of coins, a 29 Zorkmid coin and 70 Zorkmid coin.

29 **70**

Is it possible to buy, with exact change, a single Zorkoni candy, which is worth just 1 Zorkmid? If so, how can it be done?

♣ *No surprise, we need to solve Euclid's coin problem.*

$$\begin{aligned} -12: \quad 70 &= 2 \cdot 29 + 12 \\ +5: \quad 29 &= 2 \cdot 12 + 5 \\ -2: \quad 12 &= 2 \cdot 5 + 2 \\ 1: \quad 5 &= 2 \cdot 2 + 1 \end{aligned}$$

So $\gcd(70, 29) = 1$, *and* $(-12)70 + (29)29 = 1$ *So can pay for just one by giving 29 of the 29 Zorkmids, and getting 12 of the 70 zorkmids as change.* ♣

3. Let $A = \{0, 1, 2\}$ and $B = \{2, 3, 4\}$. Compute each of the following.

You may you expressions like 5^7, $\binom{7}{5}$, or $\left\{ {7 \atop 5} \right\}$ in you answers.

a) How many functions are there with domain $\mathcal{P}(A \cup B)$ and target $\mathcal{P}(\mathcal{P}(B))$?

♣ *By the multiplicative principle, the domain has cardinality* $2^5 = 32$ *and the target has cardinality* $2^8 = 256$. *So the number of functions is* $(2^8)^{(2^5)} = 256^{32}$. ♣

b) How many onto functions are there with domain $\mathcal{P}(A \cup B)$ and target $\mathcal{P}(A \cap B)$?

♣ *The target set* $A \cap B$ *has cardinality 1, so* $|\mathcal{P}(A \cap B)| = 2^1 = 2$, *we have* $\left\{ {8 \atop 2} \right\} 2!$. ♣

c) How many onto functions are there with domain $\mathcal{P}(B)$ and target A?

♣ *This should look familiar:*

$$\binom{3}{3}3^8 - \binom{3}{2}2^8 + \binom{3}{1}1^8 - \binom{3}{0}0^8 = \left\{ {8 \atop 3} \right\} 3!$$

♣

d) How many onto functions are there with domain $\mathcal{P}(B)$ and target $\mathcal{P}(A)$ if the elements of the target are are regarded as indistinguishable?

♣ *The cardinalities are the same, the onto functions are also one-to-one, and if the domain or target is indistinguishable, there is only 1 function: In this case, each element of the domain is assigned to something. Whichever formula you use, you should get 1.* ♣

4. Let X be a set. Suppose that every prime number is an element of X and that there exists a function $f : X \to \mathbb{N}$ which is one-to-one.

Label each of the following C for countably infinite, U for uncountable, I for indeterminable.

____ X

____ $X \times \mathbb{Q}$

____ $X \cap \mathbb{R}$

____ $\mathcal{P}(X \cap \{n^2 \mid n \in \mathbb{N}\})$

____ $X \cup \mathbb{R}$

♣ *The prime elements insure that* X *is an infinite set, and one-to-one function insures that* X *is countably infinite, so the first and second are TRUE.* $X \cap \mathbb{R}$ *is an infinite subset of a countably infinite, set, so is countably infinite. The fourth set might be finite, or even empty, so we have to choose* X. *The last set contains an uncountable set, so is uncountable,* U. ♣

5. Prove carefully by induction that for all $n \geq 0$ we have

$$1 + 9\sum_{k=0}^{n} 10^k = 10^{n+1}.$$

♣ *Induction might seem a roundabout way to show that* $1 + 999,999 = 1,000,000$*, but why not?*

Base Case: For $n = 0$ *we have* $1+9\cdot 10^0 = 1+9 = 10 = 10^1$*, so the base case is proved.*

Induction step. Suppose the result is true for n*. We compute*

$$\begin{aligned} 1 + 9\sum_{k=0}^{n+1} 10^k &= \left[1 + 9\sum_{k=0}^{n} 10^k\right] + 9\cdot 10^{n+1} \\ &= 10^{n+1} + 9\cdot 10^{n+1} \quad \textit{by the induction hypothesis} \\ &= (1+9)10^{n+1} = 10^{n+2} \end{aligned}$$

so the statement is also true for $n+1$*, and so the result is true for all* $n \geq 0$ *by induction.* ♣

6. Let p, q and r be statements. Label each of the following as T if it must be true, F if it must be false, and X if it cannot be determined.

____ $p \Rightarrow (p \vee q \vee r)$.

____ $[\neg(p \Rightarrow q)] = [p \vee \neg q]$

____ $\neg p \Rightarrow p$

____ $p \Rightarrow [p \vee ((q \Rightarrow r) \vee (r \Rightarrow q))]$.

____ $[p \wedge (q \wedge \neg r)] \Rightarrow [p \vee (q \wedge \neg q)]$.

♣ *If you check these with truth tables, it takes forever, but if you focus on what the statements mean, they are all very simple.*

First is T *since* p *true implies* p *OR anything whatsoever.*

Second is F *since the implication is an OR statement,* $q \vee \neg p$*, and the negation of an OR statement is an AND statement, not an OR statement by Demorgan's laws. (So the equality of the statements if* F*, but we gave credit for* X *as well.)*

The third is TRUE if p *is true, and FALSE if* p *is false. (Thank you Mr. Oates)*

The fourth is T *for the same reason as the first.*

The fifth is T*, and essentially the same as the first and fourth: assuming the antecedent assumes* p *AND other things are true. If* p *is true then* p *OR anything whatsoever is also true, so the consequence would then be true.*

♣

7. How many six letter words either start repeat after two letters, like *bobobo*, or repeat after three letters, like *booboo*, or read the same forwards and backwards, like *boppob*.

♣ *By now these problems should be very easy, and your writeup should include the definitions of the sets involved. Let* A *be those that repeat after* 2, B *those that repeat after* 3, *and* C *be those that are the same frontwards or backwards.*

The multiplicative principle helps us count all the parts: $|A| = 26^2$, $|B| = 26^3$ *and* $|C| = 26^3$. *Pairwise,* $|A \cap B| = 26$, *since all characters must be the same,* $|A \cap C| = 26$ *for the same reason, and* $|B \cap C| = 26^2$, *and so* $|A \cap B \cap C| = 26$.

Now inclusion exclusion gives:

$$|A \cup B \cup C| = |A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C| = 26^2+26^3+26^3-26-26-26^2+26.$$

♣

8. a) How many anagrams of $POLYTECHNIC$ have the letters of $YELP$ occurring in that order, such as $\underline{Y}CC\underline{E}HTIN\underline{L}O\underline{P}$?

Justify your answer.

♣ *This was a earlier quiz problem.* $\binom{11}{4}$ *gives the positions of* $YELP$. *Then* $7!$ *ways to order the other characters, with two different types of* C, *say red and blue. Then dividing by two removes the color over-count. So* $(\binom{11}{4} \cdot 7!)/2$. ♣

b) How many anagrams of $POLYTECHNIC$ have the letters of $YELP$ occurring as a substring, like $CC\underline{YELP}HTINO$?

Justify your answer.

♣ *Again just the multiplicative principle: You first have to choose the position of the* Y, *which has* 8 *choices, and forces the positions of* ELP. *There are* $7!$ *ways to order the other* 7 *characters, with distinguishing two types of* C, *then dividing by two handles the over-counts:* $(8 \cdot 7!)/2$. ♣