Ma2201/CS2022 **Discrete Mathematics** D Term, 2016

FINAL EXAM

PRINT NAME NEATLY: ____________________

$\mathcal{SIGN}$: ____________________

PLEASE DO ANY 6 of the following 8 problems. Work carefully and neatly.

1. (5 pts) A tech company currently has 5 programmers, 3 engineers, 1 accountant, and 12 administrators.

A programmer hiring committee requires two programmers, two engineers and four administrators.

An engineering hiring committee requires three engineers, one programmer, and four administrators.

An administrator hiring committee consists only administrators.

For the following, an algebraic expression is sufficient.

a) How many different administrator hiring committees can be formed?

♣: Every subset but the empty set could be a committee, so $2^{12} - 1$

b) How many different programmer hiring committees can be formed?

♣: Three independent choices of two of five programmers $\binom{5}{2}$, two of three engineers $\binom{3}{2}$ and four of twelve administrators, $\binom{12}{4}$, so $\binom{5}{2}\binom{3}{2}\binom{12}{4}$ by the multiplicative principle.

c) Suppose the company needs to hire one engineer and one programmer and doesn't want anyone to be a member of both committees. How many ways can that be done?

♣: 0, there aren't enough engineers.

2. (5 pts) Suppose a_n is a sequence of numbers with $a_0 = 2$, $a_1 = 10$, and the remaining elements defined recursively by

$$a_{n+1} = 5a_n - 6a_{n-1}$$

a) Find a formula for a_n.

♣: It is a linear recursion, so we look for a solution of the form Ar^n, so $Ar^{n+1} = 5Ar^n - 6Ar^{n-1}$ gives the characteristic equation $0 = r^2 - 5r + 6 = (r-2)(r-3)$, with solution $A2^n + B3^n$.

$a_0 = 2$ gives $A + B = 2$ and $a_1 = 10$ gives $2A + 3B = 10$, so $B = 6$ and $A = -4$.

So $a_n = (-4)2^n + 6 \cdot 3^n = -2^{n+2} + 2 \cdot 3^{n+1}$

b) Compute $\lim_{n\to\infty} \frac{a_n}{e^n}$

♣: $\lim_{n\to\infty} \frac{2^n}{e^n} = 0$ and $\lim_{n\to\infty} \frac{3^n}{e^n} = \infty$, so the limit is ∞.

3. (5 pts) Let A and B be two sets. Use the double inclusion method to show $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

♣: We first show $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$. Let $x \in \overline{A \cap B}$, so $x \notin A \cap B$, so either $x \notin A$ or $x \notin B$. That is to say, $x \in \overline{A}$ or $x \in \overline{B}$, hence $x \in \overline{A} \cup \overline{B}$ as required.

We next show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$. Let $x \in \overline{A} \cup \overline{B}$, so $x \in \overline{A}$ or $x \in \overline{B}$. If $x \in \overline{A}$, then $x \notin A$, so $x \notin A \cap B$, so $x \in \overline{A \cap B}$. If $x \in \overline{B}$, then $x \notin B$, so $x \notin A \cap B$, so $x \in \overline{A \cap B}$. So in either case $x \in \overline{A \cap B}$, as required.

Since $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$, the result is true.

4. (5 pts) Prove by induction that $n^2 + 7n - 7$ is odd for all $n \geq 0$.

♣:

Base Case: $n = 0$. The statement is that $0^2 + 7 \cdot 0 - 7 = -7$ is odd, which is true.

Induction Step: Assume the Induction Hypothesis, that $n^2 + 7n - 7$ is odd. So $n^2 + 7n - 7 = 2k + 1$ for some k.

Consider $(n + 1)^2 + 7(n + 1) - 7$. We have

$$(n+1)^2 + 7(n+1) - 7 = n^2 + 2n + 1 + 7n + 7 - 7 = (n^2 + 7n - 7) + 2n + 8 = 2k + 1 + 2n + 8$$

by the induction hypothesis, s

$$(n + 1)^2 + 7(n + 1) - 7 = 2(k + n + 4) + 1$$

and $(n + 1)^2 + 7(n + 1) - 7$ is odd, as required.

Since the base case is true and the induction step is satisfied, the statement is true for all $n \geq 0$ by induction.

5. (5 pts) Let p, q and r be statements.

a) Write $(p \Rightarrow q) \Rightarrow r$ in conjunctive normal form.

♣: Using the definition of $\Rightarrow$,

$$(p \Rightarrow q) \Rightarrow r = r \vee \neg(p \Rightarrow q) = r \vee \neg(q \vee \neg p) = r \vee (\neg q \wedge p)$$

which is in disjunctive normal form, which is the answer to part b). Using the distributive law we have

$$r \vee (\neg q \wedge p) = (r \vee \neg q) \wedge (r \vee p)$$

which is in conjunctive normal form.

b) Write $(p \Rightarrow q) \Rightarrow r$ in disjunctive normal form.

♣: $r \vee (\neg q \wedge p)$

6. (5 pts) Let A and B be finite sets, and suppose the $|A \cup B| = 100$ and $|A \cap B| = 50$.

a) What is the largest value that $|A|$ can have? What can you say about B in this case?

♣: $|A|$ cannot be larger than $|A \cup B|$, so the largest $|A|$ can be is 100, in which case B is a subset of A.

b) What is the smallest value that $|A|$ can have? What can you say about B in this case?

♣: $|A|$ cannot be smaller than than $|A \cup B|$, so the smallest $|A|$ can be is 50, in which case A is a subset of B.

c) How many one to one functions are there from $A \cap B$ to $A \cup B$?

♣: We have a formula for this. To reason it out, there are 50 choices to be made from the 100 elements of $A \cup B$, with each choice decreasing the number of options for the next choice by 1, so

$$100(99)(98)\cdots(51) = (100!)/(50!)$$

7. (5 pts) Suppose $x \equiv 6 \mathrm{mod} = 19$ and $x \equiv 7 \bmod 29$. Find x.

♣: This is the Chinese Remainder Theorem: We use the Euclidean Algorithm to find λ and μ so that $\lambda \cdot 19 + \mu \cdot 29 = 1$.

$$\begin{array}{rcll} 29 & = & 1 \cdot 19 + 10 & : 2 \\ 19 & = & 1 \cdot 10 + 9 & : -1 \\ 10 & = & 1 \cdot 9 + 1 & : 1 \end{array}$$

Adding gives $(2)(29) + (-3)(19) = 1$. (Checking $58 - 57 = 1$)

So $(6)(2)(29)$ is 6 modulo 19 and 0 modulo 29.

Similarly $(7)(-3)(29)$ is 7 modulo 29 and 0 modulo 19.

Adding them gives the result: $(6)(2)(29) + (7)(-3)(19) = 348 - 399 = -51$. To make it positive we add $(19)(29) = 551$ and get $x = 500$.

8. (5 pts) Compute $2^{2222222} \bmod 101$

♣: By Fermat's Little Theorem, $2^{100} \equiv 1 \bmod 101$, since 101 is prime. So $2^{2222222} = 2^{2222200} 2^{22} = (2^{100})^{22222} 2^{22} \equiv 2^{22} \bmod 101$ For 2^{22} we use fast multiplication:

$$2^{22} = 4^{11} = 4 \cdot 4^{10} = 4 \cdot 16^5 = 4 \cdot 16 \cdot 16^4 = 4 \cdot 16 \cdot (16^2)^2$$

so we still have to do 4 stupid multiplications.

$16^2 = 256 = 54 \bmod 101$

$54^2 = 2916 \equiv 2310 \equiv 88 \bmod 101$

So $4 \cdot 16 \cdot (16^2)^2 \equiv 4 \cdot 16 \cdot 88 \equiv 64 \cdot 88 \equiv 5632 \equiv 77 \bmod 101$

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