Exercises for Lectures 3 and 4

Note: In these exercises the complement of A is denoted by A^c, and $\mathbb{N} = \{1, 2, 3, \ldots\}$.

1. Which of the following sets are well defined.

   [Hint: You can show a set is not well defined if you can exhibit an object for which membership can not be determined from the specification.]

   a) The set numbers.

   b) The set of words in the bible.

   c) The set of rocks on the moon.

   d) The set of characters in Tolkein's book, the Hobbit.

   e) The set real numbers whose decimal representation does not have two identical digits in a row.

   f) The set of pieces in the game of chess.

   g) The set of characters in Shakespeare's "Hamlet".

   h) The set numbers.

   i) The set of museums in Washington D.C.

   j) The set of keys on a piano.

   k) The set of immortal rabbits.

   For each example which you determine is not a set, find appropriate qualifiers to make a well defined set.

2. For each of the following, write a description of the set in bracket notation.

   a) The set of prime numbers

   b) The set of integers whose final digit is 3.

   c) The set of integers which are perfect squares.

   d) The set of real numbers whose distance from π on the number line is less than $1/e$.

   e) The set of irrational numbers.

3. Let $E = \{n \in \mathbb{Z} \mid n = 2k \text{ for some } k \in \mathbb{Z}\}$,

   $O = \{n \in \mathbb{Z} \mid n \notin E\}$,

   $X = \{n \in \mathbb{Z} \mid -10 \leq n \leq 10\}$,

   Describe in bracket notation $E \cup X$, $O \cap X$, $E \cap O$, and $E \cup (X \cap O)$.

4. Let $A = \{a \in \mathbb{N} \mid a = 2i \text{ for some } i \in \mathbb{Z}\}$,

   let $B = \{b \in \mathbb{N} \mid b = 3j \text{ for some } j \in \mathbb{Z}\}$,

   and let $C = \{c \in \mathbb{N} \mid c = 6k \text{ for some } k \in \mathbb{Z}\}$,

   Use the double inclusion method to show that $A \cap B = C$.

5. Let $A = \{n \in \mathbb{N} \mid n = 5k \text{ for some } k \in \mathbb{Z}\}$,

   let $B = \{n \in \mathbb{N} \mid n = 7k \text{ for some } k \in \mathbb{Z}\}$,

   and let $C = \{n \in \mathbb{N} \mid n = 35k \text{ for some } k \in \mathbb{Z}\}$,

   Use the double inclusion method to show that $A \cap B = C$.

6. Show that for any sets A, B and C that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

7. Show that for any sets A, B and C that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

8. Let A, B, and C, be sets of integers, and let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5, 7\}$.

   $(A \cap B) \cup C = B$

   What is the largest number of elements that C can have?

   What is the least number of elements that C can have?

9. Let the universe $U = \{a, b, c, \ldots, z\}$. How many sets are in this universe?

   What is $\{a, c, e\} \cup \{j, a, c, k\}^c$?

   What is $(\{a, c, e\} \cup \{k, i, n, g\})^c$?

   What is $(\{d, o, g\}^c \cap \{c, a, t\}^c)^c$?

   What is $(\{b, i, k, e\} \cap (\{d, o, g\} \cup \{d, o, g\}^c))^c$?

   What is $(\{b, i, k, e\} \cup (\{d, o, g\} \cap \{d, o, g\}^c))^c$?

10. Let the universe $U = \{n \in \mathbb{N} \mid n \leq 2^{100}\}$.

    How many sets are in this universe?

    Let $A = \{n \in U \mid n = k^2, k \in U\}$.

    $B = \{n \in U \mid n = k^{20}, k \in U\}$.

    What is $A \cap B$.

    What is $A \cup B$.

11. Let A and B be sets with universe U. Prove $(A \cup B)^c = A^c \cap B^c$

12. Let A and B be sets with universe U. Prove $(A \cap B)^c = A^c \cup B^c$ in two ways. First directly by showing separately that $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A \cap B)^c \subseteq A^c \cup B^c$; second by using the result of the previous exercise.

13. Let $H = \{0, 1\}$. Compute $\mathcal{P}(H)$, $\mathcal{P}(\mathcal{P}(H))$, and $\mathcal{P}(H) \cap \mathcal{P}(\mathcal{P}(H))$,

14. Let $A = \{\emptyset, 0\}$ and $B = \{\{\}, \{\emptyset, \{\emptyset\}\}\}$, and $C = \emptyset$. Compute $\mathcal{P}(A \cup B \cup C)$.

    Compute $\mathcal{P}(A \cap B \cap C)$.

15. Let $A = \{\emptyset, a\}$ and $B = \mathcal{P}(A) \cup \{a\}$, and $C = \mathcal{P}(B) \cup \{a\}$

    Compute $A \cup B \cup C$.

    Compute $A \cap B \cap C$.