Ma2201/CS2022 Final Exam

Discrete Mathematics

D Term, 2013 SOLUTIONS:

ENJOY YOUR SUMMER!

1. (**10 points**) Let $A = \{1, 2, 3, \ldots, 10\}$, let $B = \{1, 2, 3, \ldots, 2^{10}\}$ and let $C = \mathcal{P}(A)$ be the power set of A. Take the universe to be the set of all integers, and sets of integers.

Mark each of the following statements with T if it *must* be true, F if it *must* be false, X no conclusion can be drawn.

___ $A \cap B \cap C = \emptyset$.
___ $A \cap B \cap C = \{\emptyset\}$.
___ $(A \cap B) \cup C = (A \cup C) \cap (A \cap C)$.
___ $\{|C|\} \subseteq B$
___ $B \in C$
___ $A \in C$
___ $A \cap B \subseteq C$
___ $(A \cup B \cup C)^c \cap B = \emptyset$
___ $|C \cup \emptyset \cup \{\emptyset\}| = |B|$
___ $(A \cap B)^c = A^c \cup B^c$

$\underline{\ T\ }$ $A \cap B \cap C = \emptyset$.

Elements of A and B are integers, elements of C are sets of integers, their intersection is empty.

$\underline{\ F\ }$ $A \cap B \cap C = \{\emptyset\}$.

$\emptyset$ is not an integer, $\emptyset \notin A$.

$\underline{\ F\ }$ $(A \cap B) \cup C = (A \cup C) \cap (A \cap C)$.

Since $(A \cap C) \subseteq (A \cup C) \cap (A \cap C)$, we have $(A \cup C) \cap (A \cap C) = (A \cap C) \subseteq A$, But $(A \cap B) \cup C \supseteq C$, so since there are elements of C not in A, the equation is false.

$\underline{\ T\ }$ $\{|C|\} \subseteq B$

$|C| = |\mathcal{P}(A)| = 2^{|A|} = 2^{10}$.

$\underline{\ F\ }$ $B \in C$

$B \notin C = \mathcal{P}(A)$ since $B \not\subset A$.

$\underline{\ T\ }$ $A \in C$

$A \subseteq A$.

$\underline{\ T\ }$ $A \cap B \subseteq C$

$A \cap B = A \subseteq A$.

$\underline{\ T\ }$ $(A \cup B \cup C)^c \cap B = \emptyset$

$(A \cup B \cup C)^c \cap B = A^c \cap B^c \cap C^c \cap B$ and $B^c \cap B = \emptyset$.

$\underline{\ T\ }$ $|C \cup \emptyset \cup \{\emptyset\}| = |B|$

$C \cup \emptyset = C$ for any C and since $\{\emptyset\} \subseteq C$, $C \cup \{emptyset\} = C$. So $|C \cup \emptyset \cup \{\emptyset\}| = |C| = 2^{|A|} = 2^{10} = |B|$.

$\underline{\ T\ }$ $(A \cap B)^c = A^c \cup B^c$

This is De Morgan's Law. It is true regardless of A, B, and C.

2. (**10 points**) Suppose that p_n is a statement, $n = 1, 2, 3, \ldots$, and suppose $p_n \Longrightarrow p_{n+2}$ for all n, and suppose that $(p_3 \vee p_7) \Rightarrow p_6$

Circle each of the following which must be true.

a) $p_3 \Longrightarrow p_6$	b) $p_3 \vee p_6 \vee p_7$
c) $p_5 \Longrightarrow p_6$	d) $p_5 \Longrightarrow p_5$
e) $p_6 \vee \sim(p_3 \vee p_7)$	f) $(p_6 \vee \sim p_3) \wedge (p_6 \vee \sim p_7)$
g) p_{100}	h) p_{1001}
i) $p_1 \Longrightarrow p_{1000}$	j) $\sim p_5 \Longrightarrow \sim p_3$

[Note that all the requirements would be satisfied if all the p_i's were false, or, in the other extreme, if all the p_i's were true.]

a) $p_3 \Longrightarrow p_6$

Since $p_3 \Longrightarrow p_3 \vee p_7 \Longrightarrow p_6$.

b) $p_3 \vee p_6 \vee p_7$

Can be false by the note above.

c) $p_5 \Longrightarrow p_6$

Since $p_5 \Longrightarrow p_7 \Longrightarrow p_3 \vee p_7 \Longrightarrow p_6$.

d) $p_5 \Longrightarrow p_5$

$p_5 \Longrightarrow p_5$ *is equivalent to* $p_5 \vee \sim p_5$ *which is always true.*

e) $p_6 \vee \sim(p_3 \vee p_7)$

$p_6 \vee \sim(p_3 \vee p_7)$ *is equivalent to* $(p_3 \vee p_7) \Rightarrow p_6$

f) $(p_6 \vee \sim p_3) \wedge (p_6 \vee \sim p_7)$

From e) use DeMorgan's law and the distributive law: $p_6 \vee \sim(p_3 \vee p_7) = p_6 \vee (\sim p_3 \wedge \sim p_7) = (p_6 \vee \sim p_3) \wedge (p_6 \vee \sim p_7)$

g) p_{100}

Can be false by the note above.

h) p_{1001}

Can be false by the note above.

i) $p_1 \Longrightarrow p_{1000}$

$p_1 \Longrightarrow p_3 \Longrightarrow p_3 \vee p_7 \Longrightarrow p_6$, *and the inductive step implies all even statements beyond* $n = 6$, *including* p_{1000}.

j) $\sim p_5 \Longrightarrow \sim p_3$

Is equivalent to $p_3 \Longrightarrow p_5$ *which is the* $p_n \Longrightarrow p_{n+2}$ *for* $n = 3$.

3. (**10 points**) Let $U = \{1, 2, 3, \ldots, 1200\}$. Let $A \subseteq U$ consist of those elements which are even. Let Let $B \subseteq U$ consist of those elements which are evenly divisible by 3. Let $C \subseteq U$ consist of those elements which are evenly divisible by 5.

Compute $|A \cup B \cup C|$ and compute $|A \cap B \cap C|$.

$|A| = 1200/2 = 600$.
$|B| = 1200/3 = 400$.
$|B| = 1200/5 = 240$.
The elements of $A \cap B$ *are divisible by 6:* $|A \cap B| = 1200/6 = 200$.
The elements of $A \cap C$ *are divisible by 10:* $|A \cap C| = 1200/10 = 120$.
The elements of $B \cap C$ *are divisible by 15:* $|B \cap C| = 1200/15 = 80$.
The elements of $A \cap B \cap C$ *are divisible by 30:* $|A \cap B \cap C| = 1200/30 = 40$.
So the total by inclusion exclusion is

$$600 + 400 + 240 - 200 - 120 - 80 + 40 = 880$$

4. (**10 points**) Prove that for all n

$$\sum_{i=0}^{n} \binom{n}{i} \binom{2n}{n+i} = \binom{3n}{n}$$

See the solutions to quiz 7

5. (**10 points**) Let $A = (0,1)$, $B = \mathcal{P}(A)$, $C = \mathcal{P}(B)$, and let D be the set of all functions with domain A and target B.

a) Compute the number of one to one functions with domain B and target D.

b) Compute the number of onto functions with domain D and target C.

c) Compute the number of onto functions with domain D and target B.

d) Compute the number of onto functions with domain D and target A.

$|A| = 2$, $|B| = 2^2 = 4$, $C = 2^4 = 16$ *and* $|D| = |B|^{|A|} = 4^2 = 16$.

a) We have to choose distinct images for each of the four domain values: $16 \cdot 15 \cdot 14 \cdot 13$

b) They have equal cardinalities, so every onto function is one to one. $16!$. *(Inclusion/exclusion also works.)*

c) The number of non-onto functions is

$$\binom{4}{3} \cdot 3^{16} - \binom{4}{2} \cdot 2^{16} + \binom{4}{1} \cdot 1^{16} - \binom{4}{0} \cdot 1^{16}$$

So the total is

$$\binom{4}{4} \cdot 4^{16} - \binom{4}{3} \cdot 3^{16} + \binom{4}{2} \cdot 2^{16} - \binom{4}{1} \cdot 1^{16} + \binom{4}{0} \cdot 1^{16}$$

d) Since the target has only two elements, we can count directly. There are 2^{16} *functions and only* 2 *are not onto: So there are* $2^{16} - 2$ *onto functions.*

6. (**10 points**) Let A, B and C be sets. Prove that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

[Venn diagrams are not a proof]

Let $x \in A \cap (B \cup C)$. *Then* $x \in A$ *and either* $x \in B$ *or* $x \in C$.

Case 1: If $x \in C$, *then, since* $x \in A$, *we have* $x \in A \cap C$, *hence* $x \in (A \cap B) \cup (A \cap C)$, *as required.*

Case 1: If $x \in B$, *then, since* $x \in A$, *we have* $x \in A \cap B$, *hence* $x \in (A \cap B) \cup (A \cap C)$, *as required.*

In either case $x \in (A \cap B) \cup (A \cap C)$ *so*

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$