Ma1023 | Calculus III | A Term, 2013

Pseudo-Final Exam | Print Name: Pancho Bosphorus

1. Mark the following **T** and **F** for false, and **X** if it cannot be determined from the given information.

____ $1^\infty = \frac{0}{0} = 1$.

False. 1 is a number, the other expressions are invalid.

____ If $\dfrac{d}{d\theta}r(\theta_0) = 0$ then the graph of $r(\theta)$ has a horizontal tangent at θ_0 in the xy-plane.

It has a horizontal tangent in the $r\theta$-plane.

____ $\cos\left(1 + \dfrac{1}{\infty}\right) = 1$.

False. ∞ is not a number.

____ $r(\theta) = \sin(\theta)\cos(\theta)\tan(\theta)$ has as horizontal tangent at the origin in the xy plane.

True with no calculations since $r(0) = 0$.

____ If $\{a_n\}_{n=1}^\infty$ converges to π then $\sum_{n=1}^\infty a_n$ cannot converge to π.

True. The series diverges by the divergence test.

____ The series $\sum_{n=1}^\infty 17$ is a geometric series.

True. The ratio is 1.

____ Every telescoping series converges.

False. $\sum_{n=1}^\infty [n-(n-1)]$ telescopes.

____ The limit comparison test fails if the limit is 1.

False. The ratio test fails if the the limit is 1.

____ If $a_n > 0$ and $\lim\limits_{n\to\infty} a_n = 0$ then $\sum_{n=1}^\infty (-1)^n a_n$ converges by the Alternating Series Test.

False. There is one other condition to be checked.

____ The power series $\sum_{n=1}^\infty (-1)^n \frac{n^{(x-2)}}{n!}$ has radius of convergence 1.

False. It is not a power series.

____ Every Taylor Series converges at its center.

True.

____ The 5'th Maclauren Polynomial of $f(x) = |x|$ is actually a polynomial of degree 4.

False. $f(x)$ is not differentiable at 0 so there is no Maclauren Polynomial.

____ If $\lim_{n\to\infty} R_n(x) \leq 1$, then the Taylor series converges.

Can't tell. The limit would have to be zero.

____ If vector is directed into the 4'th quadrant then its magnitude is negative,

The magnitude is always positive.

____ For all vectors $\mathbf{A}\cdot\mathbf{B} = -\mathbf{B}\cdot\mathbf{A}$.

False. The dot product is commutative.

____ The plane $(1,2,1)\cdot\mathbf{X} = -1$ has distance 1 from the origin.

False. It isn't -1 either.

2. Determine for which values of x the following series converges absolutely, converges conditionally, or diverges. Show all steps required to carefully present your work.

$$\sum_{n=0}^{\infty} \frac{n^3}{1+n^4}(x+1)^n$$

Also compute the radius of convergence.

$\sum_{n=0}^{\infty} \frac{n^3}{1+n^4}(x+1)^n$ Power series with center $a=-1$.

Absolute Value Series $\sum_{n=0}^{\infty} \frac{n^3}{1+n^4}|x+1|^n$

Ratio Test: $\lim_{n\to\infty} \frac{\left(\frac{(n+1)^3}{1+(n+1)^4}|x+1|^{n+1}\right)}{\left(\frac{n^3}{1+n^4}\cdot |x+1|^n\right)} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \left(\frac{1+n^4}{1+(n+1)^4}\right)|x+1|$

$= 1\cdot 1\cdot |x+1| = |x+1|$

Sidebar: $\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \lim_{x\to\infty}\left(\frac{x+1}{x}\right) \overset{\text{l'Hopital's Rule}}{=} \lim_{x\to\infty}\frac{1}{1} = 1$

$\lim_{n\to\infty} \frac{1+n^4}{1+(n+1)^4} = \lim_{x\to\infty} \frac{1+x^4}{1+(x+1)^4} \overset{\text{l'H Rule}}{=} \lim_{x\to\infty} \frac{4x^3}{4(x+1)^3} = \lim_{x\to\infty} \frac{4}{4}\left(\frac{x}{x+1}\right)^3 = 1$

So Series Converges absolutely for $|x+1|<1$,

$-1 < x+1 < 1$

$-2 < x < 0$

Diverges for $|x+1|>1$

Test Fails for $|x+1|=1$

Endpoints: $x=0$ $\sum_{n=0}^{\infty} \frac{n^3}{1+n^4}(0+1)^n = \sum_{n=0}^{\infty} \frac{n^3}{1+n^4}$

Limit Comparison Test with $\sum_{n=1}^{\infty} \frac{1}{n}$ which diverges

$\lim_{n\to\infty} \frac{\left(\frac{n^3}{1+n^4}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^4}{1+n^4} = \lim_{x\to\infty} \frac{x^4}{1+x^4} \overset{\text{l'Hopital's Rule}}{=} \lim_{x\to\infty} \frac{4x^3}{4x^3} = 1$

The limit exists and is not 0, so since $\sum \frac{1}{n}$ diverges, so does $\sum_{n=0}^{\infty} \frac{n^3}{1+n^4}$

It is a series of positive terms so it converges absolutely.

$x=-2$ $\sum_{n=0}^{\infty} \frac{n^3}{1+n^4}(-2+1)^n = \sum_{n=0}^{\infty} \frac{n^3}{1+n^4}(-1)^n$.

Alternating Series Test $\ast\ \frac{n^3}{1+n^4} \geq 0$ Series alternates

$\ast\ f(x) = \frac{x^3}{1+x^4}$, $f'(x) = \frac{(1+x^4)3x^2 - x^3\cdot 4x^3}{(1+x^4)^2}$

$= \frac{x^2(3-x^4)}{(1+x^4)^2} \leq 0$

for $x \geq 2$,

so $\frac{n^3}{1+n^4}$ is eventually decreasing.

$\ast\ \lim_{n\to\infty} \frac{n^3}{1+n^4} = \lim_{x\to\infty} \frac{x^3}{1+x^4} = \lim_{x\to\infty} \frac{3x^2}{4x^3} = \lim_{x\to\infty} \frac{3}{4x} = 0$

interpolation l'H's Rule

So the series converges by the alternating series test.

Since the abs value series diverges, it converges conditionally.

Summary: absolute convergence $-2 < x < 0$

divergence $x \geq 0$, $x < -2$

conditional convergence $x = -2$

Radius of Convergence $R=1$

3. Determine for which values of x the following series converges absolutely, converges conditionally, or diverges. Show all steps required to carefully present your work.

$$\sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

Also compute the radius of convergence.

In the interval of converge define $f(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{2^n n!}$

For the above series, answer the following:

a) Find $f'''(0)$.

b) Compute $f'(x)$ inside the radius of convergence.

c) Compute $\int f(x)\, dx$ inside the radius of convergence.

Ratio Test: $\lim_{n\to\infty} \left(\frac{\left(\frac{|x|^{3(n+1)}}{(n+1)!}\right)}{\left(\frac{|x|^{3n}}{n!}\right)} \right) = \lim_{n\to\infty} \frac{|x|^3}{n+1} = 0$

which is less than 1, so the series converges absolutely for all x by the ratio test.

a) $f'''(0)$. The x^3 term $(n=1)$ is $\frac{x^3}{1!} = \frac{f^{(3)}(0)\,x^3}{3!}$

so $f^{(3)}(0) = 6$.

b) $f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{x^{3n}}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{x^{3n}}{n!} = \sum_{n=1}^{\infty} \frac{3n\, x^{3n-1}}{n!}$

(derivative of the constant term is 0)

c) $\int f(x)\,dx = \int \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}\,dx = \sum_{n=0}^{\infty} \int \frac{x^{3n}}{n!}\,dx = C + \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)\,n!}$

4.

a) Find the limit of the sequence $\left\{\dfrac{n^2}{2^n}\right\}$.

Show all steps required to carefully present your work.

b) Decide whether or not the series $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{2^n}$ is geometric. If it is geometric, and converges, determine the sum.

Show all steps required to carefully present your work.

c) Using the Taylor Series for $\sin(x)$ with center 0, find the Taylor series for $\sin(3x^2)$.

Show all steps required to carefully present your work.

a) $\displaystyle\lim_{n\to\infty} \frac{n^2}{2^n} \overset{\text{Interpolate}}{=} \lim_{x\to\infty} \frac{x^2}{2^x} \overset{\text{l'H's Rule}}{=} \lim_{x\to\infty} \frac{2x}{2^x \ln(2)} \overset{\text{l'H's Rule}}{=} \lim_{x\to\infty} \frac{2}{2^x(\ln(2))^2} = 0$

b) $\displaystyle\sum \frac{n^2}{2^n}$

Ratio: $\displaystyle\frac{a_{n+1}}{a_n} = \frac{\left(\frac{(n+1)^2}{2^{n+1}}\right)}{\left(\frac{n^2}{2^n}\right)} = \left(\frac{n+1}{n}\right)^2 \frac{1}{2}$

Since the Ratio is not constant the series is not geometric, and nothing else need be done

c) $\displaystyle\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ for all x

so $\displaystyle\sin(3x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (3x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1} x^{4k+2}}{(2k+1)!}$

5. Let $\mathbf{u} = 3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = 1\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$.

a) Find the magnitudes of $\mathbf{u}$ and $\mathbf{v}$ as well as unit vectors which point in their directions.

b) Find the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

c) Find $(\mathbf{u} + 2\mathbf{v}) \cdot (2\mathbf{u} + \mathbf{v})$.

d) Which of the following products is undefined, and why?

$(2\mathbf{u}) \cdot \mathbf{u}, \quad (2\mathbf{u}) \times \mathbf{u}, \quad (\mathbf{u} \cdot \mathbf{v}) \cdot \mathbf{v}, \quad (\mathbf{u} \cdot \mathbf{v}) \times \mathbf{v}, \quad (\mathbf{u} \times \mathbf{v}) \times \mathbf{v}, \quad (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v}.$

d) Find the equation of the plane which is normal to $\mathbf{u}$ and whose closest distance to the origin is $|\mathbf{v}|$.

a) $|\vec{u}| = \sqrt{9+9+9} = 3\sqrt{3}$

$|\vec{v}| = \sqrt{1+9+4} = \sqrt{14}$

$\frac{\vec{u}}{3\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

$\frac{\vec{v}}{\sqrt{14}} = \left(\frac{1}{\sqrt{13}}, \frac{-3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}\right)$

b) $\vec{u} \cdot \vec{v} = 3 - 9 + 6 = 0$

$= |u||v| \cos(\theta)$

So $\cos(\theta) = 0$

$\theta = 90°$

c) $(\vec{u} + 2\vec{v})(\vec{u} + 2\vec{v})$

$= \vec{u}\cdot\vec{u} + 4\vec{u}\cdot\vec{v} + 4\vec{v}\cdot\vec{v}$

$= 27 + 0 + 4 \cdot 14$

$= 83$

d) $(\vec{u}\cdot\vec{v})\cdot v$ is undefined.

a scalar dot a vector.

$(\vec{u}\cdot\vec{v}) \times \vec{v}$ is not defined

a scalar cross a vector.

e) plane which is normal to $\vec{u}$

and has distance $|\vec{v}|$ from origin

is

$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \cdot \vec{x} = \sqrt{14}.$

6. Let $\mathbf{f}(t) = t\mathbf{i} + t^2\mathbf{j} - t^2\mathbf{k}$ be a position vector at time t.

a) Find all values t for which the speed is 0.

b) Show that the velocity vector at $t = 0$ is a unit vector.

c) Find a unit vector tangent to the curve at $t = 1$.

d) Find any vector perpendicular to both $\mathbf{f}(1)$ and $\mathbf{v}(1)$.

$\vec{f}(t) = t\vec{i} + t^2\vec{j} - t^2\vec{k}$

a) Velocity vector $\frac{d\vec{f}(t)}{dt} = 1\vec{i} + 2t\vec{j} - 2t\vec{k}$

$\left|\frac{d\vec{f}}{dt}\right| = \sqrt{1 + 4t^2 + 4t^2} = \sqrt{1+8t^2}$

which is never 0, so the speed is never 0.

b) $\vec{v}(0) = (1, 0, 0)$

$|(1,0,0)| = \sqrt{1^2 + 0 + 0} = 1$

c) The velocity vector is tangent to the curve:

$\vec{v}(1) = (1, 2, -2)$

$|\vec{v}(1)| = \sqrt{1+4+4} = \sqrt{9} = 3$

So the unit vector is $(\frac{1}{3}, \frac{2}{3}, \frac{-2}{3})$.

d) We can just take $\vec{f}(1) \times \vec{v}(1)$

$= (1, 1, -1) \times (1, 2, -2)$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{vmatrix} = (-2+2)\vec{i} + (-1+2)\vec{j} + (2-1)\vec{k}$$

$= +\vec{j} + \vec{k}$

e) There is no e. This test was to easy.