Ma1023 | **Calculus III** | A Term, 2013

Final Exam (B) | Print Name: ____________________

1. (6 pts) For any 10 of the following, mark with **T** and **F** for false, and **X** if it cannot be determined from the given information.

a. ____ If $\mathbf{u} \cdot \mathbf{v} = 0$ then $\mathbf{u} \times \mathbf{v} = 0$.

False. Try $\mathbf{i}$ and $\mathbf{j}$.

b. ____ The limit comparison test fails if the limit is 1.

False. The ratio test fails if the the limit is 1.

c. ____ If the power series $\sum_{n=0}^{\infty} c_k (x-10)^k$ converges absolutely at $x = 0$, then it converges absolutely for $x = 15$.

True. The radius of convergence is at least 10.

d. ____ The dot product of any 2-dimensional vector with a 3-dimensional vector is a 5-dimensional vector.

False. It is undefined. And the dot product is always a scalar.

e. ____ If $f^{(5)}(5) = 5$ then $\dfrac{5(x-5)^5}{5!}$ is a term in its Taylor series centered at 5.

True.

f. ____ The equation $\mathbf{n} \cdot \mathbf{v} = 1$ defines a plane only if the normal vector $\mathbf{n}$ is a unit vector.

False. The distance to the origin is 1 only if it is a unit vector.

2. (8 pts) For each value of x determine whether the following power series converges absolutely, converges conditionally, or diverges. Show all steps required to carefully present your work.

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}(x-2)^n$$

$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n\,3^n}$ Absolute Value Series $\sum_{n=1}^{\infty} \frac{|x-2|^n}{n\,3^n}$

Ratio Test: $\lim_{n\to\infty} \frac{\left(\frac{|x-2|^{n+1}}{(n+1)3^{n+1}}\right)}{\left(\frac{|x-2|^n}{n\,3^n}\right)} = \lim_{n\to\infty} \frac{|x-2|}{3}\left(\frac{n}{n+1}\right)$

$= \lim_{n\to\infty} \frac{|x-2|}{3}\left(\frac{1}{1+\frac{1}{n}}\right) = \frac{|x-2|}{3}(1) = \frac{|x-2|}{3}$

So the series converges absolutely if $\frac{|x-2|}{3} < 1$

or $|x-2| < 3$, $-3 < x-2 < 3$

or $-1 < x < 5$.

Diverges for $x > 5$ or $x < -1$

Endpoints: $x=5$ $\sum_{n=1}^{\infty} \frac{(5-2)^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{3^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ Harmonic Series. Diverges.

$x=-1$ $\sum_{n=1}^{\infty} \frac{(-1-2)^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Does not converge absolutely (see $x=5$)

Alternating Series Test

* $\frac{1}{n} > 0$ Series alternates
* $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2} < 0$ $\frac{1}{n}$ decreasing
* $\lim_{n\to\infty} \frac{1}{n} = 0$

Series converges by the Alternating Series Test

Converges Conditionally at $x=-1$

Summary: Converges absolutely $-1 < x < 5$

Converges conditionally $x=-1$

Diverges $x \geq 5$, $x < -1$

3. (4 pts) The power series $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ converges absolutely for all x, so define

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$$

Answer each of the following:
a) Find $f^{(5)}(0)$.

b) Find the Taylor Series for $f'(x)$ centered at 0. What is its radius of convergence?

a) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!}$ Taylor Series $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)(x-a)^k}{k!}$

5th term $\frac{f^{(5)}(0)x^5}{5!} = \frac{x^{2\cdot 2+1}}{2!} = \frac{x^5}{2!}$

term in x^5

when $2n+1=5$, $n=2$

$f^{(5)}(0) = \frac{5!}{2!} = 5\cdot4\cdot3 = 60$

b) $\frac{d}{dx}\sum_{n=0}^{\infty} \frac{x^{2n+1}}{n!} = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^{2n+1}}{n!}\right) = \sum_{n=0}^{\infty} \frac{(2n+1)x^{2n}}{n!}$

(note no constant term to discard)

4. (4 pts) Using the known Taylor Series for e^x with center 0, find the Taylor series for the function $f(x) = x^2e^{2x+1}$ centered at 0.

Show all steps required to carefully present your work.

Recall $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$x^2e^{2x+1} = x^2e^{2x}e = x^2\left(\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}\right)e = \sum_{k=0}^{\infty} \frac{x^2 2^k x^k e}{k!}$

$= \sum_{k=0}^{\infty} \frac{e 2^k x^{k+2}}{k!}$

5. (8 pts) Let $\mathbf{f}(t) = \cos\left(\frac{\pi}{2}t^2\right)\mathbf{i} + \sin\left(\frac{\pi}{2}t^2\right)\mathbf{j} + \frac{\pi}{2}t^2\mathbf{k}$ be a position vector at time t.

a) Find all times where the speed is 0.

b) Show that the angle between the velocity vector $\mathbf{v}(t)$ and the vector $\mathbf{k}$ is constant for all $t > 0$.

c) Find a unit vector tangent to the curve at $t = 2$.

d) Compute $|\mathbf{f}(2) \times \mathbf{v}(2)|$.

$\vec{f}(t) = \cos(\frac{\pi}{2}t^2)\vec{i} + \sin(\frac{\pi}{2}t^2)\vec{j} + \frac{\pi}{2}t^2\vec{k}$

a) $\vec{v}(t) = -\pi t\sin(\frac{\pi}{2}t^2)\vec{i} + \pi t\cos(\frac{\pi}{2}t^2)\vec{j} + \pi t\vec{k}$

$\frac{ds}{dt} = |\vec{v}(t)| = \sqrt{\pi^2t^2\sin^2(\frac{\pi}{2}t^2) + \pi^2t^2\cos^2(\frac{\pi}{2}t^2) + \pi^2t^2}$

$= \pi|t|\sqrt{\sin^2(\frac{\pi}{2}t^2) + \cos^2(\frac{\pi}{2}t^2) + 1}$

$= \pi|t|\sqrt{1+1} = \pi\sqrt{2}|t|$

which is 0 only when $t = 0$

b) $\vec{v}(t)\cdot\vec{k} = |\vec{v}||\vec{k}|\cos(\theta)$ $\quad \vec{k} = (0,0,1)$

$\vec{v}(t)\cdot\vec{k} = \pi t$

$\cos(\theta) = \frac{\pi t}{|\vec{v}(t)||\vec{k}|} = \frac{\pi t}{(\pi\sqrt{2}t)(1)} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ $\quad \theta = \frac{\pi}{4}$

for all $t > 0$

[Note for $t<0$, $|t| = -t$, $\cos(\theta) = -\frac{\sqrt{2}}{2}$, $\theta = \frac{3\pi}{4}$]

c) $\vec{v}(2)$ is the tangent vector when $t=2$

$\vec{v}(2) = -2\pi\sin(\frac{\pi}{2}\cdot 4)\vec{i} + 2\pi\cos(\frac{\pi}{2}\cdot 4)\vec{j} + 2\pi\vec{k}$

$= 2\pi\vec{j} + 2\pi\vec{k}$

$|\vec{v}(2)| = \pi\sqrt{2}|2| = 2\sqrt{2}\pi$

Unit vector $\frac{\vec{v}(2)}{|\vec{v}(2)|} = \frac{2\pi}{2\pi\sqrt{2}}\vec{j} + \frac{2\pi}{2\sqrt{2}}\vec{k} = \frac{\sqrt{2}}{2}\vec{j} + \frac{\sqrt{2}}{2}\vec{k}$

d) $\vec{f}(2)\times\vec{v}(2) = (\vec{i} + 2\pi\vec{k})\times(2\pi\vec{j} + 2\pi\vec{k})$

$\begin{vmatrix} i & j & k \\ 1 & 0 & 2\pi \\ 0 & 2\pi & 2\pi \end{vmatrix}$ $\quad = -4\pi^2\vec{i} - 2\pi\vec{j} + 2\pi\vec{k}$

magnitude: $\sqrt{(4\pi^2)^2 + (2\pi)^2 + (2\pi)^2}$

$= 4\pi^2\sqrt{4\pi^2+1+1} = 4\pi^2\sqrt{4\pi^2+2}$

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